THE MATHEMATICAL GAZETTE

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.

AND

PROF. E. T. WHITTAKER, M.A., F.R.S.

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The Mathematical Association.

THE ANNUAL MEETING will be held at the LONDON DAY TRAINING COLLEGE, Southampton Row, London, W.C. 1, on *Monday*, 5*th January*, 1925 (Advanced Section); on *Tuesday*, 6*th January*, 1925 (Ordinary Meeting).

THE suggestion has been made that a few of the most interesting books in the Library should be brought to the Annual Meeting for display. Any member who would like to inspect any particular volume without the responsibility of borrowing it is asked to write to the Librarian as soon as possible.

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The Mathematical Association.

PUBLICATIONS.

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THE USE OF VECTORS IN PRACTICAL MATHEMATICS

BY L. B. BENNY, M.A.

THE uses of vectors are many and varied. The value of the vector notation and of vector methods of calculation is becoming recognised in mechanics and physics, but few teachers use them to any extent in teaching pure mathematics, although they appear to be equally useful in this sphere.

I have found vector methods especially useful for students who require a "practical" knowledge; they are well adapted to numerical work, and often provide a welcome change from the drudgery of memorising formulae, and calculating results by substitution of numerical values of the symbols—a dull task, lacking in educational value, which forms such a large part of the work of the technical student. Vector solutions are usually short and simple, and proceed from first principles; they should thus be preferred on logical grounds.

In what follows, I have endeavoured to indicate a possible minimum course of vector calculus for students taking a course of the type generally provided for the Ordinary and Advanced National Certificates in Engineering, and to give a few of the applications to elementary mathematics; they are merely suggestive of what can be done, and in one or two cases (given to illustrate the range of the method) are possibly beyond the requirements of the purely technical student. For economy of space the work is expressed very shortly in a form suitable for the teacher; symbols and diagrams are allowed to explain themselves whenever possible.

The work in vectors should be taught gradually, with constant numerical and graphical exercises, and might be spread over two years, the vector products being deferred until the second year. During the first year, the student should become accustomed to the new ideas involved, and should obtain ample practice in their use, especially if the teaching is correlated with his work in drawing and mechanics. Drawing exercises should include the finding of vector sums, mean vectors, mean centres of sets of points, mass-centres of systems of mass-points, and of various types of rectilinear figures, by replacing constituent triangles by equivalent masses at their corners. The applications to mechanics, which are not included here, will form a valuable complement to this work. When a student finds he is using addition and resolution of vectors in his force problems, scalar products in connexion with work, and vector products in dealing with moments of forces, he will begin to realise the power of the tool that has been placed in his hands.

A Suggested Minimum Course of Vector Calculus.

I. Vector: its definition, magnitude (tensor), unit of direction, or unit vector (ort). Notation: $\alpha = ai$, $a = |\alpha| =$ tensor, $i =$ ort. Representation by directed segment of line, $\overline{AB} = \alpha$, $AB = |\alpha|$. Like vectors,

$$m\alpha = m(ai) = ma \, . \, i.$$

Displacement and area as vectors.

II. Addition, subtraction, and resolution of vectors, especially in two or three mutually perpendicular directions, connexion between tensor of resolute and projection, resolute of vector sum equals sum of resolutes of vectors. If $l\alpha + m\beta = l'\alpha + m'\beta$, then $l = l'$, $m = m'$. Standard expression of vector in form, $\alpha = \alpha_x + \alpha_y = a_x i + a_y j$, or $= \alpha_x + \alpha_y + \alpha_z = a_x i + a_y j + a_z k$, i, j, k being fundamental orts in mutually perpendicular directions Ox, Oy, Oz, forming a R.H. system. Ort λ in direction $\hat{\theta}$ to i in plane ij, given by

$$\lambda = i \cos \theta + j \sin \theta.$$

III. Fixation of points by position vectors w.r.t. base point O. If $\overline{OP} = \alpha$, position vector of P, then $\alpha = xi + yj$ or $= xi + yj + zk$, where (x, y) or (x, y, z) are co-ordinates of P. Mean vector $\bar{\alpha} = \frac{\Sigma \alpha_1}{n}$, mean centre of set of points its unique position independent of O.

$$\overline{PQ} = \overline{PO} + \overline{OQ} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k.$$

A good deal of numerical work should be given in connexion with this section.

IV. Scalar product $\alpha\beta = |\alpha| . |\beta| . \cos(\widehat{\alpha\beta})$, geometrical interpretation, $m\alpha . n\beta = mn\alpha\beta$, $\alpha\beta = \beta\alpha$, meaning of $\alpha\beta = 0$, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$, hence distribution of products as in scalar algebra.

$$\alpha^2 = \alpha\alpha = |\alpha|^2; \qquad i^2 = j^2 = k^2 = 1, \qquad ij = jk = ki = 0.$$

$$\alpha\beta = (a_x i + a_y j + a_z k)(b_x i + b_y j + b_z k) = a_x b_x + a_y b_y + a_z b_z.$$

Multiplication by i, j, k respectively is equivalent to resolving. Deduction of two or three scalar relations from one vector relation.

V. Vector product $[\alpha\beta] = |\alpha| . |\beta| . \sin(\widehat{\alpha\beta}) . \lambda$, where λ is suitable ort, geometrical interpretation. $[\alpha\beta] = -[\beta\alpha]$, meaning of $[\alpha\beta] = 0$. $[\alpha\alpha] = 0$. $[ii] = [jj] = [kk] = 0$. $[ij] = -[ji] = k$, etc. $[m\alpha . n\beta] = mn[\alpha\beta]$.

$$[\alpha\beta] = [(a_x i + a_y j + a_z k)(b_x i + b_y j + b_z k)] = \Sigma(a_y b_z - a_z b_y)i,$$

with numerical examples for practice.

Meaning of triple product $\alpha[\beta\gamma]$, its geometrical interpretation as a volume.

A Few Typical Applications.

A. *Pure Geometry.* $\triangle ABC$, D, E bisecting AB, AC. If $\overline{AB} = \alpha$, $\overline{AC} = \beta$, $\overline{BC} = \overline{BA} + \overline{AC} = \beta - \alpha$, $\overline{DE} = \overline{DA} + \overline{AE} = \frac{1}{2}(\beta - \alpha)$, and therefore DE is $\parallel$ BC (like vectors), and $DE = \frac{1}{2}BC$.

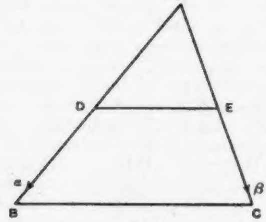


FIG. 1.

Generally, if $AD:AB=AE:AC=m$, $\overline{AD}=m\alpha$, $\overline{AE}=m\beta$, whence $\overline{DE}=m(\beta-\alpha)$, so that DE is $\parallel BC$, and $DE:BC=m$.

Conversely, given $DE \parallel BC$, $\overline{DE}$, $\overline{BC}$ are like vectors, and

$$\overline{DE}=m\overline{BC}=m(\beta-\alpha).$$

Also $\overline{AD}=p\overline{AB}=p\alpha$, $\overline{AE}=q\beta$, and $\overline{DE}=\overline{DA}+\overline{AE}$, gives $m(\beta-\alpha)=q\beta-p\alpha$, so that $q=p=m$, and $AD:AB=AE:AC=m$.

2. To prove that opposite sides of a parallelogram are equal, and that the diagonals bisect each other.

Let $\overline{AB}=\alpha$, $\overline{AD}=\beta$, then $\overline{BC}=k\beta$, $\overline{DC}=l\alpha$ (like vectors).

$$\overline{AB}+\overline{BC}+\overline{CD}+\overline{DA}=0, \text{ gives } \alpha+k\beta-l\alpha-\beta=0,$$

or $\alpha(1-l)+\beta(k-1)=0$; $\therefore$ $k=1$, $l=1$, whence $BC=AD$, $DC=AB$.

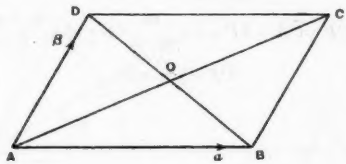


FIG. 2.

Again, $\overline{AC}=\alpha+\beta$; $\therefore$ $AO=p(\alpha+\beta)$, and $\overline{BD}=\beta-\alpha$; $\therefore$ $\overline{OD}=q(\beta-\alpha)$.
Then $\overline{AO}+\overline{OD}+\overline{DA}=0$ gives $p(\alpha+\beta)+q(\beta-\alpha)-\beta=0$;

$$\therefore \alpha(p-q)+\beta(p+q-1)=0;$$

$$\therefore p-q=0,\ p+q-1=0 \text{ or } p=q=\tfrac{1}{2}, \text{ whence } O \text{ bisects } AC \text{ and } BD.$$

If the figure is a rhombus, $AB=AD$, whence we may write $\alpha=ai$, $\beta=aj$, i, j being orts, so that $\overline{AC}=a(i+j)$, $\overline{DB}=a(i-j)$.

$$\therefore \overline{AC}\,.\,\overline{DB}=a^2(i^2-j^2)=0;$$

$$\therefore AC \text{ is perpendicular to } DB.$$

3. $\triangle ABC$, right-angled at C.

$$\gamma=\alpha+\beta;\ \therefore\ \gamma^2=(\alpha+\beta)^2=\alpha^2+\beta^2, \text{ since } \alpha\beta=0.$$

$$\therefore AB^2=BC^2+CA^2. \quad \text{(Pythagoras.)}$$

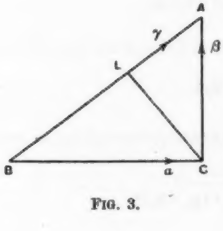


FIG. 3.

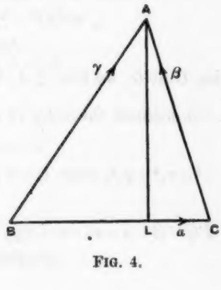


FIG. 4.

If CL is $\perp AB$, $a^2 = aa = a(\gamma - \beta) = a\gamma$;

$$\therefore\ BC^2 = BL \,.\, BA,$$

and similarly $$AC^2 = AL \,.\, AB.$$

4. ABC is any triangle, $\gamma = a + \beta$, $\gamma^2 = a^2 + \beta^2 + 2a\beta$, gives

$$c^2 = a^2 + b^2 + 2a\beta,$$

whence $$c^2 = a^2 + b^2 - 2a \,.\, CL \quad \text{if } \hat{C} \text{ is acute (diagram)},$$

and $$c^2 = a^2 + b^2 + 2a \,.\, CL \quad \text{if } \hat{C} \text{ is obtuse.}$$

Result $2a\beta = c^2 - a^2 - b^2$ is sometimes useful; writing $\overline{CB} = a$, instead of $\overline{BC}$, it becomes $2a\beta = a^2 + b^2 - c^2$, giving at once $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$.

5. If $AP : PB = m : n$, $\overline{AP} = \dfrac{m}{m+n}\ \overline{AB} = \dfrac{m}{m+n}(\beta - a)$;

$$\therefore\ \overline{CP} = \overline{CA} + \overline{AP} = a + \frac{m}{m+n}(\beta - a) = \frac{na + m\beta}{m+n}.$$

If P bisects AB, $$\overline{CP} = \tfrac{1}{2}(a + \beta).$$

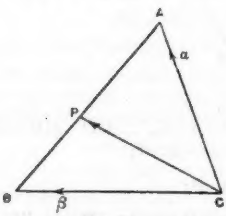


FIG. 5.

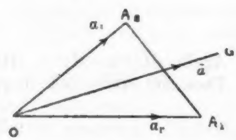


FIG. 6.

To find the length of the median, square the last result.

$$\overline{CP}^2 = \tfrac{1}{4}(a^2 + \beta^2 + 2a\beta); \quad \therefore\ CP^2 = \tfrac{1}{4}(CA^2 + CB^2 + 2a\beta),$$

which, with $2a\beta = BC^2 + CA^2 - AB^2$, gives $CP^2 = \tfrac{1}{4}(2CA^2 + 2CB^2 - AB^2)$, the method being used in numerical work, not the formula.

The general case yields as readily

$$(m+n)^2 CP^2 = m(m+n)BC^2 + n(m+n)CA^2 - mn \,.\, AB^2.$$

6. G is the mean centre of points $A_1, \dots A_n$. $n\bar{a} = \sum a_r$ gives on squaring

$$n^2 OG^2 = \sum OA_r^2 + 2\sum a_r a_s, \quad \text{and} \quad 2a_r a_s = OA_r^2 + OA_s^2 - A_r A_s^2;$$

$$\therefore\ n^2 OG^2 = \sum OA_r^2 + \sum(OA_r^2 + OA_s^2 - A_r A_s^2);$$

$$\therefore\ n^2 OG^2 = n\sum OA_r^2 - \sum A_r A_s^2;$$

putting O at G, we find $\sum A_r A_s^2 = n\sum GA_r^2$.

B. *Co-ordinate Geometry* (2 *dimensions*).

$$a = x_1 i + y_1 j, \quad \beta = x_2 i + y_2 j.$$

1. $a^2 = x_1^2 + y_1^2$, since $ij = 0$;

$$\therefore\ OP^2 = x_1^2 + y_1^2.$$

2. $\overline{PQ} = \beta - a = \overline{x_2 - x_1}\, i + \overline{y_2 - y_1}\, j$;

$$\therefore \text{ squaring, } PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

3. $\alpha\beta = x_1x_2 + y_1y_2$;

$$\therefore\ r_1r_2\cos P\hat{O}Q = x_1x_2 + y_1y_2$$

and
$$P\hat{O}Q = \cos^{-1}\left\{\frac{x_1x_2 + y_1y_2}{\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}}\right\}.$$

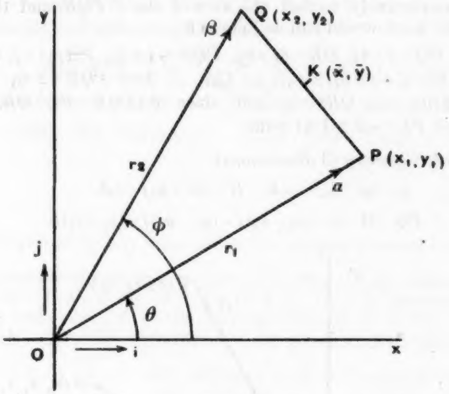


FIG. 7.

Or 4.
$$\cos(\phi - \theta) = \frac{x_1x_2}{r_1r_2} + \frac{y_1y_2}{r_1r_2}$$
$$= \cos\phi\cos\theta + \sin\phi\sin\theta;$$

changing the sign of θ,

$$\cos(\phi + \theta) = \cos\phi\cos\theta - \sin\phi\sin\theta,$$

giving an exceedingly simple general proof of addition formula for cosines.

5.
$$[\alpha\beta] = (x_1y_2 - x_2y_1)k, \text{ where } k \text{ is ort} \perp xOy.$$
$$\therefore\ 2\Delta OPQ = (x_1y_2 - x_2y_1).$$

Or 6.
$$r_1r_2\sin(\phi - \theta) = x_1y_2 - x_2y_1,$$
$$\sin(\phi - \theta) = \frac{x_1y_2}{r_1r_2} - \frac{x_2y_1}{r_1r_2} = \sin\phi\cos\theta - \cos\phi\sin\theta,$$

and changing the sign of θ,

$$\sin(\phi + \theta) = \sin\phi\cos\theta + \cos\phi\sin\theta.$$

7. If $K(\bar{x}, \bar{y})$ divides PQ in the ratio $m : n$,

$$\overline{OK} = \frac{n\alpha + m\beta}{m+n},$$

whence on scalar multiplication by i and j,

$$\bar{x} = \frac{nx_1 + mx_2}{m+n}, \quad \bar{y} = \frac{ny_1 + my_2}{m+n}.$$

8. In the same way we find, at once, for the mean centre of a set of points,

$$\bar{x}=\frac{\Sigma x_1}{n}, \quad \bar{y}=\frac{\Sigma y_1}{n}.$$

9. In every numerical case, the work proceeds on the lines indicated, and no formula need be quoted. As an example, if P, Q, R are the points (2, 1), (1, 4), (3, 2) respectively, to find the area of the $\triangle PQR$, and the length of $PL \perp QR$. The work would run as follows:

$$\overline{OP}=2i+j,\ \overline{OQ}=i+4j,\ \overline{OR}=3i+2j,\ \overline{PQ}=-i+3j,\ \overline{PR}=i+j,\ \overline{QR}=2i+2j.$$

$$[\overline{PQ}\,.\,\overline{PR}]=[(-i+3j)(i+j)]=-4k;\quad \therefore \text{ Area } PQR=2 \text{ sq. units.}$$

$\overline{QR}^2=(2i-2j)^2$; $\therefore QR^2=8$, and since $2\Delta PQR=PL\,.\,QR$, we have $4=2\sqrt{2}PL$, and $PL=\sqrt{2}\simeq 1{\cdot}41$ units

C. *Co-ordinate Geometry* (3 *dimensions*).

$$a=x_1i+y_1j+z_1k, \quad \beta=x_2i+y_2j+z_2k.$$

$$\overline{PQ}=\beta-a=(x_2-x_1)i+(y_2-y_1)j+(z_2-z_1)k.$$

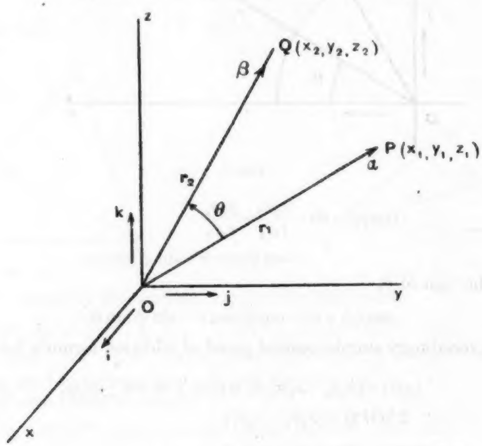


FIG. 8.

1. $a^2=x_1^2+y_1^2+z_1^2$, or $OP^2=x_1^2+y_1^2+z_1^2$.

2. $\overline{PQ}^2=\Sigma\overline{x_2-x_1}^2$, or $PQ^2=\Sigma(x_2-x_1)^2$.

3. $a\beta=x_1x_2+y_1y_2+z_1z_2$,

giving $\qquad \cos\theta=\dfrac{x_1x_2+y_1y_2+z_1z_2}{r_1r_2}$

$\qquad\qquad =l_1l_2+m_1m_2+n_1n_2$ in notation of direction-cosines.

4. $[a\beta]=\Sigma(y_1z_2-y_2z_1)i$;

$\therefore\ 2\Delta OPQ\,.\,\lambda=\Sigma(y_1z_2-y_2z_1)i$, λ being ort $\perp$ plane OPQ;

and squaring, $\qquad 4(\Delta OPQ)^2=\Sigma(y_1z_2-y_2z_1)^2$,

and $\qquad$ Area $OPQ=\frac{1}{2}\sqrt{\Sigma(y_1z_2-y_2z_1)^2}$.

Or 5. $r_1 r_2 \sin\theta \,.\, \lambda = \sum (y_1 z_2 - y_2 z_1) i$, whence squaring,

$$\sin^2\theta = \sum\left(\frac{y_1 z_2 - y_2 z_1}{r_1 r_2}\right)^2 = \sum (m_1 n_2 - m_2 n_1)^2.$$

6. If R is any third point (x_3, y_3, z_3), $\overline{PQ} = \sum (x_2 - x_1) i$, $\overline{PR} = \sum (x_3 - x_1) i$;

$$\therefore\ 2\Delta PQR \,.\, \lambda = [\overline{PQ} \,.\, \overline{PR}] = \sum \{\overline{y_2 - y_1}\,\overline{z_3 - z_1} - \overline{y_3 - y_1}\,\overline{z_2 - z_1}\} i,$$

whence squaring and extracting the square root,

$$\Delta PQR = \tfrac{1}{2}\sqrt{\sum \{\overline{y_2 - y_1} \,.\, \overline{z_3 - z_1} - \overline{y_3 - y_1} \,.\, \overline{z_2 - z_1}\}^2}.$$

7. If V is volume of $\|$piped with OP, OQ, OR as adjacent edges,

$$V = [\overline{OP} \,.\, \overline{OQ}] \,.\, \overline{OR} = \sum (y_1 z_2 - y_2 z_1) i \,.\, \sum x_3 i = \sum x_3 (y_1 z_2 - y_2 z_1)$$

$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix},$$

and if O is replaced by any other point, the volume is found in the same way.

8. As a numerical example, suppose OP, OQ, OR are adjacent edges of a $\|$piped, where P, Q, R are (3, 1, 1), (1, 4, 2), (2, 1, 5) respectively, and we require the inclination, θ, of the diagonal through O to OP. If OK is this diagonal, $\overline{OK} = \overline{OP} + \overline{OQ} + \overline{OR} = 6i + 6j + 8k$, whence $OK^2 = 136$ and

$$\overline{OP} = 3i + j + k;\ \therefore\ OK \,.\, OP \cos\theta = \overline{OK} \,.\, \overline{OP} = 18 + 6 + 8 = 32;$$

$$\therefore\ \cos\theta = \frac{32}{\sqrt{11 \times 136}} = \frac{16}{\sqrt{374}} \simeq 0{\cdot}8273, \text{ giving } \theta.$$

Municipal Technical School,
Leigh, Lancs. L. B. BENNY.

GLEANINGS FAR AND NEAR.

270. "Even more severe was the sentence passed (by Valentinian and Valens, A.D. 365 ? *Cod. Theod.* xvi. 5. 3) against the hapless Mathematicians. In words which would now carry terror through the pleasant places by the Cam, the Imperial brothers decreed: 'Let the discourse of the Mathematicians cease. For if in public or in private, by night or by day, anyone shall be caught instructing another in this forbidden error, both the teacher and taught shall be sentenced to capital punishment. For it is no less a crime to teach than to learn forbidden arts.' By Mathematicians were doubtless here meant Astrologers. . . ."—Hodgkin: *Italy and her Invaders*, vol. i. pp. 204-5.

271. "The Theodosian Code teemed with enactments against Mathematicians, meaning of course primarily the imposters who calculated nativities and cast horoscopes."

Cf. *Cod. Theod.* ix. 16. 12. Mathematicos, nisi parati sint, Codicibus erroris proprii sub oculis Episcoporum incendio concrematis, Catholicae religionis cultui fidem tradere, nunquam ad errorem praeteritum credituri, non solum Urbe Roma, sed etiam omnibus civitatibus pelli decernimus."—*Ibid.* iii. p. 541. (Per. F. Puryer White, M.A.)

272. Swift said of Newton that when asked a question he would revolve it round and round before he could produce an answer.

TRILINEARS.

By H. V. Lowry, M.A.

In note 690 (xi. p. 390) I gave a number of trigonometrical identities which are satisfied by the angles θ, ϕ, ψ, which a straight line makes with the sides of a triangle. The whole theory of trilinears can be built up on this basis without any reference to Cartesian coordinates and without loss of symmetry.

It is first necessary to establish another group of identities.

If $l_1m_1n_1$, $l_2m_2n_2$ are any six numbers,

$$\Sigma l_1 \cos\theta \Sigma l_2 \cos\theta + \Sigma l_1 \sin\theta \Sigma l_2 \sin\theta$$
$$= \Sigma l_1 l_2 - \Sigma(m_1n_2 + m_2n_1)\cos A = J_{12}, \text{ say, } \quad\text{(1)}$$

$$(\Sigma l_1 \cos\theta)^2 + (\Sigma l_1 \sin\theta)^2 = \Sigma l_1^2 - 2\Sigma m_1 n_1 \cos A = J_1^2, \text{ say, } \quad\text{(2)}$$

and if a be the angle between two lines given by $\theta_1\phi_1\psi_1$, $\theta_2\phi_2\psi_2$:

$$\Sigma l_1 \cos\theta_1 \Sigma l_2 \sin\theta_2 - \Sigma l_2 \cos\theta_2 \Sigma l_1 \sin\theta_1 = J_{12}\sin a + I_{12}\cos a, \quad\text{(3)}$$

$$\Sigma l_1 \cos\theta_1 \Sigma l_2 \cos\theta_2 - \Sigma l_1 \sin\theta_1 \Sigma l_2 \sin\theta_2 = J_{12}\cos a - I_{12}\sin a, \quad\text{(4)}$$

where I_{12} stands for $\Sigma(m_1n_2 - m_2n_1)\sin A$.

The Equation of a Straight Line.

If $\alpha - \alpha_1/\sin\theta_1 = \beta - \beta_1/\sin\phi_1 = \gamma - \gamma_1/\sin\psi_1 = s$ and $l_1\alpha + m_1\beta + n_1\gamma = 0$ represent the same straight line, $\Sigma l_1 \sin\theta_1 = 0$; and therefore substituting in (1) and (2), $\quad \Sigma l_1 \cos\theta_1 = J_1, \quad J_1\Sigma l_2 \cos\theta_1 = J_{12}$.

For two straight lines (3) and (4) then give

$$O = J_{12}\sin a + I_{12}\cos a, \quad J_1J_2 = -I_{12}\sin a + J_{12}\cos a;$$

$$\therefore \sin a = -I_{12} \div J_1J_2, \quad \cos a = I_{12} \div J_1J_2, \quad \tan a = -I_{12} \div J_{12}.$$

The two lines will be perpendicular if $J_{12} = 0$, and parallel if $I_{12} = 0$.

The line $l_1\alpha + m_1\beta + n_1\gamma + k(a\alpha + b\beta + c\gamma) = 0$ is parallel to $l_1\alpha + m_1\beta + n_1\gamma$, since $\Sigma(l_1 + ka)\sin\theta = \Sigma l_1 \sin\theta_1 + \Sigma a \sin\theta_1 = 0$.

Putting $l_2 = 1$, $m_2 = 0$, $n_2 = 0$, in the expressions for $\cos a$ and $\sin a$,

$$\sin\theta = (m\sin C - n\sin B) \div J, \quad \cos\theta = (1 - m\cos C - n\cos B) \div J,$$

dropping the suffix 1. $\therefore$ the line through $\alpha'\beta'\gamma'$ perpendicular to

$$l\alpha + \beta m + n\gamma = 0$$

is
$$\frac{\alpha - \alpha'}{1 - m\cos C - n\cos B} = \frac{\beta - \beta'}{m - n\cos A - l\cos C} = \frac{\gamma - \gamma'}{n - l\cos B - m\cos A}.$$

The distance between two points was found in Note 690.

The perpendicular distance of $\alpha'\beta'\gamma'$ from $l\alpha + m\beta + n\gamma = 0$ is p, where p is given by the fact that the point $(\alpha' + p\cos\theta, \dots)$ must lie on the given line.

$$\therefore p\Sigma l\cos\theta = -\Sigma l\alpha'. \quad \therefore p = -\Sigma l\alpha' \div J.$$

To find the equation of the circle on $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$, as diameter:

If $\alpha\beta\gamma$ is on the circle $\alpha = \alpha_1 + s_1\sin\theta - \alpha_2 + s_2\cos\theta$ for some values of s_1 and s_2; $\therefore (\alpha - \alpha_1)^2 \div s_1^2 + (\alpha - \alpha_2)^2 \div s_2^2 = 1$. Forming the similar equations for β and γ and eliminating s_1, s_2, the equation of the circle is

$$\left.\begin{matrix} (\alpha - \alpha_1)^2, & (\alpha - \alpha_2)^2, & 1 \\ (\beta - \beta_1)^2, & (\beta - \beta_2)^2, & 1 \\ (\gamma - \gamma_1)^2, & (\gamma - \gamma_2)^2, & 1 \end{matrix}\right\} = p,$$

which becomes

$$\Sigma a\,[2b\gamma - \beta(\gamma_1 + \gamma_2) - \gamma(\beta_1 + \beta_2) + \beta_1\gamma_2 + \beta_2\gamma_1] = 0,$$

on making use of the identities $\Sigma a(\alpha - \alpha_1) = 0$, $\Sigma a(\alpha - \alpha_2) = 0$.

The General Equation of the Second Degree.

The expression $u\alpha^2+v\beta^2+w\gamma^2+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta$ will be denoted by S, and the similar function with $\sin\theta$ instead of α, etc., by S', the $'$ indicating that the expression is a function of direction and not of coordinates ; P_{12} will be used to denote $(u\alpha_1+h\beta_1+g\gamma_1)\alpha_2+(h\alpha_1+v\beta''+f\gamma_1)\beta_2+(g\alpha_1+f\beta_1+w\gamma_1)\gamma_2$; P_1 will denote P_{12} with the suffix $_2$ dropped.

The following identities for the direction functions of the second degree can be easily established.

$$S_1'S_2'-(P_{12}')^2=-k\sin^2\alpha, \qquad\qquad (5)$$

where α is the angle between the two directions and k the determinant

$$\begin{vmatrix} u & h & g & \sin A \\ h & v & f & \sin B \\ g & f & w & \sin C \\ \sin A & \sin B & \sin C & 0 \end{vmatrix}.$$

$$S'+S''=q, \qquad\qquad (6)$$

where S'' is the same function of $90+\theta$, etc., as S' is of θ, and

$$q=u+v+w-2f\cos A-2g\cos B-2h\cos C.$$

Using X, Y, Z to represent $u\alpha+h\beta+g\gamma$, etc., the line through $\alpha_1\beta_1\gamma_1$ in direction $\theta\phi\psi$ cuts the conic $S=0$ at a distance s from $\alpha_1\beta_1\gamma_1$ given by :

$$s^2S'-2s(X'\alpha_1+Y'\beta_1+Z'\gamma_1)+S_1=0. \qquad\qquad (7)$$

From this equation the equations of the tangent and polar follow easily, and are here omitted.

The centre of the conic is the point which bisects all chords through it.

$\therefore$ $\alpha_1\beta_1\gamma_1$ is the centre if (7) has equal roots.

$\therefore$ $X'\alpha_1+Y'\beta_1+Z'\gamma_1=X_1\sin\theta+Y_1\sin\phi+Z_1\sin\psi=0$ for all values of θ, ϕ, ψ.

$\therefore$ the centre is given by the equations $X_1/a=Y_1/b=Z_1/c=S_1/2\Delta=\lambda$, say.

Since the 2nd term in (7) vanishes if $\alpha_1\beta_1\gamma_1$ is the centre, the semi-diameter in direction $\theta\phi\psi$ is given by $s^2=S_1/S'=-\dfrac{\lambda}{2\Delta S'}$.

For a circle the semi-diameters in three different directions are equal.

$\therefore$ taking these directions as $(0, 180+C, 180-B)$, etc., the radius r of the circle is given by $S_1\div r^2=v\sin^2C+w\sin^2B-2f\sin B\sin C=\ldots=\ldots$, whence the usual conditions for a circle.

If the semi-diameter in direction $\theta\phi\psi$ tends to infinity it is parallel to an asymptote ; $\therefore$ the directions of the asymptotes are given by $S'=0$. The conic will be a rectangular hyperbola if S' and $S''=0$.

$\therefore$ the condition that the conic should be a rectangular hyperbola is $q=0$.

The equation $S=0$ represents two straight lines if the centre lies on the curve. The directions of the lines are given by $S'=0$, and if the lines are inclined at an angle α, $S'(\theta+\alpha, \phi+\alpha, \psi+\alpha)$ also $=0$.

$$\therefore\ \cos^2\alpha S'+\sin^2\alpha S''+2\sin\alpha\cos\alpha P'(\theta\phi\psi, 90+\theta, 90+\phi, 90+\psi)=0.$$

But from the identities (5) and (6), if $S'=0$,

$$P'(\theta\ldots, 90+\theta\ldots)=\sqrt{k}, \quad S''=q.$$

$$\therefore\ q\sin^2\alpha+2\sin\alpha\cos\alpha\sqrt{k}=0.$$

$$\therefore\ \tan\alpha=-\frac{2\sqrt{k}}{q}.$$

It follows that if the directions are those of the asymptotes of a conic, the asymptotes are real, unreal, or parallel, if k is $+$ve, $-$ve or 0.

$\therefore$ the conic is a hyperbola, ellipse or parabola, if k is $+$ve, $-$ve or 0.

To find the directions of the bisectors of the angles between the directions given by $S'=0$.

$$S'=1/2\{q-[u\cos 2\theta+\ldots+2f\cos(\phi+\psi)+\ldots]\}.$$

$$\therefore\ u\cos 2\theta+\ldots+f(\cos 2\phi+\cos 2\psi)\div\cos 2A+\ldots=q.$$

$$\therefore\ \xi\cos 2\theta+\eta\cos 2\phi+\zeta\cos 2\psi=q,\ \text{where } \xi=u+h/\cos C+g/\cos B.$$

$\therefore$ if $\theta_1\phi_1\psi_1$ and $\theta_2\phi_2\psi_2$ are the directions given by $S'=0$, and a the angle between them :

$$\Sigma\xi\cos 2\theta_1=\Sigma\xi\cos 2\theta_2=q.$$

$$\therefore\ \Sigma\xi\sin(\theta_1+\theta_2)=0\quad\text{and}\quad\Sigma\xi\cos(\theta_1+\theta_2)\cos a=q,$$

since

$$\theta_1-\theta_2=\phi_1-\phi_2=\psi_1-\psi_2=a.$$

$\therefore$ if $\theta'\phi'\psi'$ is the direction of one of the bisectors,

$$\Sigma\xi\sin 2\theta'=0\quad\text{and}\quad\Sigma\xi\sin 2A\sin 2\theta'=0\quad\text{(see Note 690).}$$

$\therefore$ $\sin 2\theta'$ is proportional to $\eta\sin 2C-\zeta\sin 2B$.

Since $\Sigma\xi\sin 2\theta'=0$, $\Sigma\xi\cos 2\theta'=\Sigma\xi^2-2\Sigma\eta\zeta\cos A\}^{\frac{1}{2}}=J(\xi\eta\zeta)$.

$$\therefore\ \cos a=q\div J(\xi\eta\zeta).$$

$\therefore$ using the value of $\tan a$, $\sin a=-2\sqrt{k}\div J(\xi\eta\zeta)$, and these two methods of finding a also give the identity $q^2+4k=J(\xi\eta\zeta)^2$.

Conjugate Diameters of a Conic.

The diameter conjugate to $la+m\beta+n\gamma=0$ with respect to $S=0$ is the polar of the point at an infinite distance in direction $\theta\phi\psi$ of the line.

$\therefore$ its equation is $\quad X\sin\theta+Y\sin\phi+Z\sin\psi=0,$

but $\quad l\sin\theta+m\sin\phi+n\sin\psi=0.$

$\therefore$ the equation of the diameter can be written in the determinant form

$$\begin{vmatrix} X & Y & Z \\ l & m & n \\ a & b & c \end{vmatrix}=0.$$

The directions $\theta_1\phi_1\psi_1$, $\theta_2\phi_2\psi_2$ will be conjugate if $P_{12}'=0$.

$\therefore$ using (5), $\quad S_1'S_2'=k\sin^2 a.$

$\therefore$ if s_1, s_2 are the lengths of the semi-conjugate diameters in the given directions,

$$s_1^2s_2^2=4\Delta^2\lambda^2\div k\sin^2 a.$$

For the axes of the conic $\theta_2=90+\theta_1$ and $a=90$.

$\therefore$ using (5) and (6), $\quad S'S''=k,\quad S'+S''=q.$

$$\therefore\ \{S'\}^2-qS'+k=0.$$

$$\therefore\ S'=1/2(q\pm\sqrt{q^2+4k})=1/2(q+J(\xi\eta\zeta)).$$

$\therefore$ the semi-axes are given by the equations

$$r^2=-4D\lambda\div q\pm J(\xi\eta\zeta)\rightarrow\frac{-4D\lambda}{q\pm J(\xi\eta\zeta)}.$$

The Equation of the Axes.

To find the equation of the axes it is necessary to prove the following geometrical proposition.

If p_1, p_2, p_3 are the perpendiculars from the centre of a conic on to any three tangents at the ends of three diameters, lengths $2s_1$, $2s_2$, $2s_3$, and d_1, d_2, d_3, the perpendiculars from any point on to the conjugate diameters, then

$$\begin{vmatrix} \dfrac{d_1^2}{p_1^2} & \dfrac{d_2^2}{p_2^2} & \dfrac{d_3^2}{p_3^2} \\ s_1^2 & s_2^2 & s_3^2 \\ 1 & 1 & 1 \end{vmatrix}=0.$$

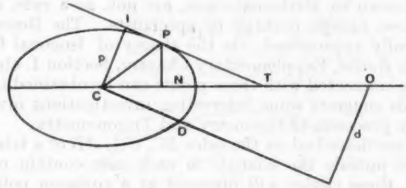
The proof given is for a point on the major axis.

Let O be any point on the major axis, d the perpendicular from it on to CD, the diameter conjugate to CP, p the perpendicular from C on to the tangent at P, and PN the perpendicular from P on to the major axis. Let the tangent at P meet the major axis in T. Then

$$d^2/p^2 = (CO/CT)^2 = CO^2 \,.\, CN^2 \div CA^2 = CO^2/CA^2\{CP^2 - CB^2 \div CA^2 - CB^2\}.$$

$\therefore$ $d^2/p^2 = \rho s^2 + \tau$, where ρ, τ are independent of the position of P.

$\therefore$ for any three diameters $d_1{}^2/p_1{}^2 = \rho s_1{}^2 + \tau$, etc.; and eliminating ρ and τ, we get the result enunciated.



Let CP_1 be in the direction $\theta_1\phi_1\psi_1$; then P is the point $(\alpha + s_1 \sin \theta_1, \ldots, \ldots)$.

$\therefore$ the equation of P_1T is $\Sigma(s_1X_1' + a\lambda)\alpha = 0$, but the equation of $C\bar{D}_1$ is

$$X_1'\alpha + Y_1'\beta + Z_1'\gamma = 0, \quad V_1 = 0, \text{ say.}$$

$$\therefore\ p_1 = 2\Delta\lambda \div s_1\bar{J} \text{ and } d_1 = V_1 \div \bar{J}, \text{ where } \bar{J} = \Sigma(X_1')^2 - 2\Sigma Y_1'Z_1' \cos A.$$

$$\therefore\ d_1/p_1 = V_1 s_1/2\Delta\lambda.$$

$\therefore$ substituting in the above result,

$$\begin{vmatrix} V_1{}^2s_1{}^2 & V_2{}^2s_2{}^2 & V_3{}^2s_3{}^2 \\ s_1{}^2 & s_2{}^2 & s_3{}^2 \\ 1 & 1 & 1 \end{vmatrix} = 0. \qquad \therefore \begin{vmatrix} V_1{}^2 & V_2{}^2 & V_3{}^2 \\ \dfrac{1}{s_1{}^2} & \dfrac{1}{s_2{}^2} & \dfrac{1}{s_3{}^2} \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

This is a very general form of the equation of the axes, since the diameters may be in any directions whatever.

Taking $CP_1, \ldots$ parallel to the sides of the triangle of reference, $1/s_1{}^2$ is proportional to $wb^2 + vc^2 - 2fbc$, and $V_1 = 0$ is $Zc - Yb = 0$.

$\therefore$ the equation of the axes is

$$\begin{vmatrix} (Zb - Yc)^2 & (Xc - Za)^2 & (Ya - Xb)^2 \\ wb^2 + vc^2 - 2fbc & uc^2 + wa^2 - 2gca & va^2 + ub^2 - 2hab \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

H. V. Lowry.

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273. My very friend Master Recorde, Doctor of Physic, singularly seen in all the Seven Sciences, and a great Divine, visited me in the prison (to his great peril if it had been known, who long time was at charges and pains with me, gratis), and also after I was delivered. By means whereof, and the Providence of God, I received my health.—*Narrative of Edward Underhill, of the Band of Gentleman Pensioners, surnamed the "Hot Gospeller"* (Harl. MS. 425).

He sent for my friend, before spoken of, Doctor Record, who examined him [Robert Allen the "Prophesier" and astrologer]: and he [Allen] knew not the rules of Astronomy; but "Was a very unlearned ass; and a sorcerer, for the which he was worthy hanging," said Master Recorde.—*Loc. cit.*

THE GEOMETRY OF THE TRIANGLE.

By H. E. Piggott.

Teachers of elementary Mathematics are generally very glad to find some unfamiliar field into which they can turn their pupils to exercise themselves in elementary processes. The following is a suggestion of a method of approaching certain theorems in connection with the Geometry of the Triangle which, although well known to Mathematicians, are not, as a rule, studied as part of a school course, except perhaps by specialists. The Brocard Points and Circle are generally approached, via the theory of Isogonal Conjugates (see Casey's *Sequel to Euclid*, Supplementary Chapter, Section I. etc.). But many of the properties connected with these points can be obtained in a more direct manner, and this suggests some interesting investigations involving nothing but the ordinary processes of Geometry and Trigonometry.

(I.) If circles are described on the sides BC, CA, AB of a triangle as chords, so that the arcs outside the triangle in each case contain respectively the angles A, B, C, these circles will intersect at a common point O, since the sum of the angles $180°-A$, $180°-B$, $180°-C$, is $360°$. This point can easily be shown to be the ortho-centre (Fig. 1).

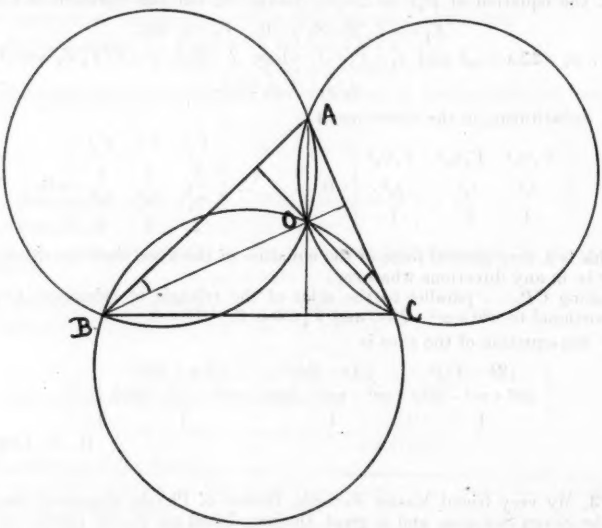

Fig. 1.

Incidentally it may be observed, that certain angles are equal in pairs, viz: OBA and OCA; OBC and OAC; OAB and OCB.

(II.) If now on the chords BC, CA, AB are described circles such that the arcs external to the triangle contain angles of C, A, B respectively these circles will also intersect at a common point (for the same reason as before). The practical construction is simplified in this case, since AC is a tangent to the circle through B and C, and similarly for the others (Fig. 2).

Suppose we denote angle PBC by θ (P being the point of intersection of the arcs). Then $P\hat{C}A = P\hat{B}C$ since AC is a tangent to the circle BPC.

Similarly $P\hat{A}B = P\hat{C}A$. Thus P is situated so that the angles PBC, PCA and PAB are all equal.

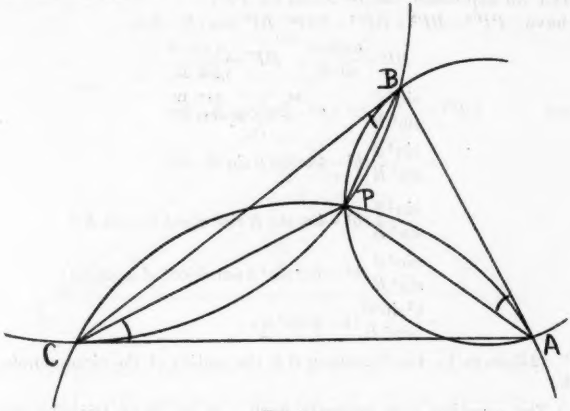

FIG. 2.

(III.) An expression for θ in terms of the angles A, B, C can easily be obtained. Using the facts $\frac{PC}{BC} = \frac{\sin\theta}{\text{sic } C}$ and $\frac{PC}{AC} = \frac{\sin(A-\theta)}{\sin A}$ it is an easy exercise in Trigonometry to obtain $\cot\theta = \cot A + \cot B + \cot C$.

(IV.) The Ortho-centre O, obtained in (I.), lies within the triangle only if this is acute-angled. The point P, however, always lies within the triangle. This is obvious from Geometrical considerations. It can be shown also from (III.). For, if C is the smallest angle of the triangle, for this to be true, $\cot\theta$ must be greater than $\cot C$, *i.e.* $\cot A + \cot B$ must be positive, which can easily be shown to be true.

(V.) It is of interest to consider the value of θ in certain cases.

(i) If $A = 90°$, $\cot\theta = \cot B + \tan B = 2 \operatorname{cosec} 2B$.

Hence (a) for the triangle 90°, 45°, 45°, $\cot\theta = 2$, $\theta = 26° \, 34'$;

(b) for the triangle 90°, 60°, 30°, $\cot\theta = \frac{4}{\sqrt{3}}$, $\theta = 23° \, 23'$

(ii) For an isosceles triangle in which $B = C$,

$$\cot\theta = \cot B + \operatorname{cosec} A.$$

(VI.) If another set of circles is described on BC, CA, AB, containing angles in the arcs external to the triangle of B, C, A respectively, these will intersect in a point P' so situated that $P'\hat{C}B = P'\hat{A}C = P'\hat{B}A$. By the symmetry of the result in (III.), it is clear that each of these angles is θ as there defined. P and P' are known as the Brocard points, and θ as the Brocard angle of the triangle.

(VII.) From Geometrical considerations it is clear that of the sets of lines AP, AP'; BP, BP'; CP, CP'; the set through the smallest angle, at all events, must overlap, *i.e.* $\theta > \frac{C}{2}$, if C is the smallest angle.

This suggests the Trigonometrical inequality that

$\cot A + \cot B + \cot C < \cot \frac{C}{2}$, where C is the smallest angle of the triangle ABC. This can easily be shown.

(VIII.) An expression can be found for PP'.

We have $PP'^2 = BP^2 + BP'^2 - 2BP \cdot BP' \cos(B - 2\theta)$.

$$BP = \frac{c \sin\theta}{\sin B}, \quad BP' = \frac{a \sin\theta}{\sin B}.$$

Hence
$$\begin{aligned} PP'^2 &= \frac{\sin^2\theta}{\sin^2 B}(c^2 + a^2 - 2ac\cos\overline{B - 2\theta}) \\ &= \frac{\sin^2\theta}{\sin^2 B}(b^2 - 4ac\sin\theta\sin\overline{B-\theta}) \\ &= \frac{\sin^2\theta}{\sin^2 B}(b^2 - 4ac\sin B\sin^2\theta\{\cot\theta - \cot B\}) \\ &= \frac{\sin^2\theta}{\sin^2 B}(b^2 - 4ac\sin^2\theta\sin B\{\cot A + \cot C\}) \\ &= \frac{b^2\sin^2\theta}{\sin^2 B}(1 - 4\sin^2\theta). \end{aligned}$$

$\therefore$ $PP' = 2R\sin\theta\sqrt{1 - 4\sin^2\theta}$, where R is the radius of the circumcircle of the triangle.

(IX.) The question now suggests itself: if we have triangles inscribed to a given circle, for what value of θ will the distance PP' be a maximum. By differentiation, $\sin\theta\sqrt{1 - 4\sin^2\theta}$ is a maximum when $\sin^2\theta = \frac{1}{8}$, whence $\theta = 20° 42'$.

If the triangle is right-angled, using V (i) we have $2 \operatorname{cosec} 2B = \sqrt{7}$ or $\sec 4B = -7$, whence $B = 24° 33'$.

For triangles inscribed to the same circle, when PP' is a maximum its length is $\frac{1}{4}$ of the diameter of the circle.

(X.) If BP, CP', be produced to meet in K, CP, AP', in L, and AP, BP', in M, since $P\hat{L}P = P\hat{K}P' = 2\theta$, and $P\hat{M}P = 180° - 2\theta$, therefore the points P, P', K, L, M, are concyclic.

(XI.) $M\hat{K}L = M\hat{P}'L = A - 2\theta + AMP' = A$.

Hence the triangle KLM is similar to the triangle ABC.

The ratio of the linear dimensions of these triangles is $LM : BC$. Using the fact that $\frac{LM}{PP'} = \frac{\sin A}{\sin 2\theta}$ the ratio of the linear dimensions comes to be

$$\sqrt{\tfrac{1}{4} - \tfrac{3}{4}\tan^2\theta}.$$

There are some interesting special cases:

(i) For the triangle whose angles are 90°, 45°, 45°, PP' is one-fifth of the hypotenuse, and the above ratio is $\frac{1}{4}$.

(ii) For the triangle 90°, 60°, 30°, the ratio is $\frac{\sqrt{7}}{8}$.

(XII.) Since $K\hat{B}C = K\hat{C}B = \theta$, K is on the perpendicular bisector of BC. Similarly L is on the perpendicular bisector of AC.

The acute angle between these perpendiculars is C. But $K\hat{M}L = C$.

Hence the circumcentre H lies on the Brocard circle as defined in (X.).

(XIII.) If through K and L, lines be drawn parallel to BC and AC respectively, intersecting in V, since $K\hat{V}L = C = K\hat{M}L$, it follows that V is on the circle, that it is the other extremity of the diameter through H, and that the line through M parallel to AB also passes through V.

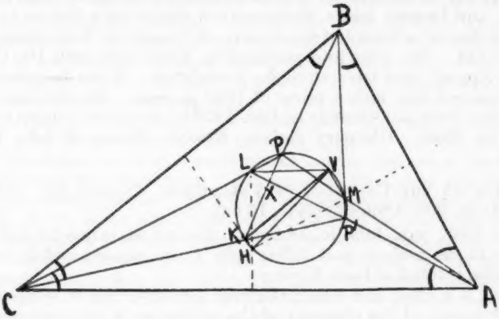

FIG. 3.

(XIV.) The following are some results which follow from the above :

(i) If AP', BP, meet in X, $AX : BX = b^2 : a^2$, and similarly.

(ii) If P and consequently P', lie on the bisector of one of the angles (A), then a is a mean proportional between b and c.

(iii) Two other symmetrical forms for θ are :

$$\sin^3\theta = \sin(A-\theta)\sin(B-\theta)\sin(C-\theta).$$

and
$$3\sin\theta = \sin(2A+\theta) + \sin(2B+\theta) + \sin(2C+\theta).$$

(iv) If ρ_1, ρ_2, ρ_3 are the radii of the circles in Fig. 2, then

$$\rho_1\rho_2\rho_3 = R^3.$$

(v) If t be the length of the tangent from A to the Brocard circle,

then
$$t_a = \frac{bc}{\sqrt{a^2+b^2+c^2}}.$$

Hence the tangents from A, B, C, to the Brocard circle are inversely proportional to the sides of the triangle.

(vi) $AP \,.\, BP \,.\, CP = AP' \,.\, BP' \,.\, CP' = (2R\sin\theta)^3$.

It does not seem profitable to pursue the investigation further on these lines. Casey shows that V is the 'symmedian' point, *i.e.* the point of concurrence of the isogonal conjugates to the medians of the triangle with respect to the angles through which they pass. He first works out the properties of isogonal conjugates and of the symmedian point, defines the Brocard circle as that on HV as diameter, and then deduces most of the properties arrived at above. The foregoing is merely suggested as a fairly simple chain of Geometrical properties which seem to grow naturally out of an obvious experiment with circles and a triangle, and which may be worth doing, either for its intrinsic interest, or as a preliminary to a more complete study of the Geometry of the Triangle.

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NOTES ON SOME BOOKS IN THE LIBRARY.*

ABBATT, Richard.

b. 1800, educated at a boarding school in Yorkshire: apprenticed to a grocer, but his indentures were cancelled owing to his ill-health. He was a Quaker, and became first a Mathematical Master at a School in Hammersmith, and then at a School of the Society of Friends, at Tottenham. Elected F.R.A.S. 1833. He went into partnership, 1836-1839, with Dr. Usmer at a school at Epping, and later at Stoke Newington. When he retired, his old pupils presented him with a purse of 1000 guineas. He was one of the expedition that went to Gibraltar in Dec. 1870 to study the eclipse of the Sun. d. Sept. 15, 1884. Obituary Notice, *Monthly Notices R.A.S.*, Feb. 1885, pp. 188-9.

A TREATISE ON THE CALCULUS OF VARIATIONS. Second ed. 1841. Thos. Ostell, xii, 208, 1 folding plate, 16 cm.

First ed. 1836, pub. John Richardson. Second ed. is line-for-line reprint of first, even to the preface, and differs only in the imprint and in being dated from London instead of from Epping.

The book is a clear and straightforward account, rich in worked examples and in references, of the elements of the subject as it was understood before Weierstrass. Only a few pages are given to multiple integrals, and there is no mention of the second variation.

The author begins by remarking that his is the first work in English on the Calculus of Variations since Woodhouse's *Isoperimetrical Problems*, 1810, to embrace the subject generally. He was followed in 1850 by J. H. Jellett, Provost of Trinity, Dublin, whose work was on a larger scale; since then the subject which Jacobi and Hamilton made the basis of Analytical Dynamics has been confined again in this country to articles in encyclopædias and chapters in books on the Integral Calculus; in America and on the Continent there has been no lack of treatises.

The Library would welcome (1) *Abbatt's other mathematical works,* which are all of a kind to interest members of the Association;* (2) *the classical treatises of Woodhouse and Jellett mentioned above;* (3) *Carll's treatise, New York,* 1881, *or London,* 1885, *for many years the standard work in the English language.*

* His other works are:
Elements of Plane and Spherical Trigonometry, 1832 and 1836.
Principles and Practice of linear perspective divested of all difficulties, 1853.
General Education: learning made easy, or teaching by reason and sight, 1854.
Orthographic projection of the globe on the plane of a given horizon: or perspective view of the earth, 1857.
A short introduction to the Principia, or the first steps in Physical Astronomy, 1868.
The Elements of Physical Astronomy, 1870.
Remarks on the Infinitesimal Calculus, 1876.
He also published two "Orthogonal Projections of the World," representing the earth floating in space.

ABBOTT, Edwin Abbott.

FLATLAND | A ROMANCE OF MANY DIMENSIONS | With Illustrations by | the Author, A Square. 1884

Seeley, viii, 100, 21½ cm. Publisher's vellum wraps, figured.

[Pres. W. J. Greenstreet

This famous book was published anonymously, but the authorship was no secret. It has not been reprinted, but a Dutch edition is said to be still in print. A careful account of the varieties of perception geometrically possible in a two-dimensional world is enlivened by social satire. The male inhabitants

The following obvious abbreviations are used:—*Pres.* = Presented by: *Rev.* = Reviewed in.

of the two-dimensional world, unless rendered Irregular by vice, have the shape of Isosceles Triangles or Regular Polygons; the less the intellect, the sharper the angle; the sharpest Triangles form the military class, the aristocracy are Circles. Women are Straight Lines, and being invisible in certain positions and preternaturally sharp, would be a perpetual source of danger but for a strict code: separate doors by which all women shall enter "in a becoming and respectful manner" (compared with separate entrances to village churches for "Villagers, Farmers, and Teachers of Board Schools"), compulsion to maintain a continual clamour and execution if subject to fits.

The first stage in the ennoblement of a family is the birth of an Equilateral from Isosceles parents; this occurrence "is welcomed not only by the poor serfs themselves, . . . but also by the Aristocracy at large, . . . well aware that these were phenomena, while they do little or nothing to vulgarise their own privileges, serve as a most useful barrier against revolution from below." In the event of sedition, "Art comes to the aid of Law and Order. It is generally found possible—by a little artificial compression or expansion on the part of the state physicians—to make some of the more intelligent leaders of a rebellion perfectly Regular, and to admit them at once into the privileged classes; a much larger number . . ., allured by the prospect of being ultimately ennobled, are induced to enter the State hospitals, where they are kept . . . for life; one or two alone . . . are led to execution. Then the wretched rabble of the Isosceles, planless and leaderless, are either transfixed without resistance by the small body of their brethren whom the Chief Circle keeps in pay . . .; or . . . by means of jealousies and suspicions skilfully fomented . . . by the Circular party, they are stirred to mutual warfare, and perish by one another's angles."

ABENBÉDER, Abi Abdalla Mohamad ben Omar.

Unknown except as the author of the following book, the editor of which has found no confirmation of any conjectures about him by historians of Arab culture in Spain.

COMPENDIO DE ALGEBRA. Texto árabe, traducción y estudio por J. A. S. Pérez. 1916. Madrid; Junta para amplicación de estudios e investigaciones científicas. xlviii, 118, 78 (Arabic texte). 2 photographic reproductions of MS. 23 cm. Pub. wraps.

[Pres. W. J. Greenstreet

The Arabic MS., in the Library of the Escurial, from which this text is transcribed, is dated anno Egiræ 744, that is, A.D. 1343, but there is no clue to the date of composition of the original work.

The author discusses simple and quadratic equations, operations with roots of numbers, and multiplication of signs, and solves illustrative problems; the editor gives not only a verbal translation of the text, but also the equivalent modern symbolical forms of the formulæ and equations.

[Rev. M.G. x, p. 95

ADAMS, John Couch, 1819-1892.

Declined knighthood, 1847; Lowndean Professor, 1858-1892 (between Peacock and Ball); Director of the Cambridge Observatory, 1861-1892 (between Challis and Ball); President R.A.S., 1851-3, 1874-6. One of the two discoverers of Neptune; his calculations were completed the earlier by seven months, but Le Verrier's were the first to be confirmed, and Challis, Plumian Professor and Director of the Observatory at Cambridge, had to defend himself and the Astronomer Royal, Airy, for allowing the discovery to be claimed first from a German observatory on behalf of a French astronomer. Sheepshanks, in a letter to Schumacher, alludes to the modesty of Adams, and continues: "I think there is a hope that Mr. Adams will continue his

astronomical researches. In any other country there could be no doubt of it, but in England . . . the Law or the Church seizes on all talent which is not independently rich or careless of wealth."

In addition to his astronomical work, which included investigations on the moon's motion that anticipated some of those of G. W. Hill, and a determination of the orbit of the November meteors, Adams calculated a number of mathematical constants to as many as 273 places of decimals; he was in fact a perfect computer, neat, accurate, and tireless, and his skill in this capacity stood him in good stead as an astronomer. He devoted much time to Newton's unpublished manuscripts, presented to the University in 1872, and his work was incorporated in a Catalogue published in 1888.

Schoolboys are familiar with the association of the name of Adams with a focal property of the conic, but it is seldom realised that the discoverer of the property was the great astronomer: indeed, not a few writers describe the theorem as Adam's.

[Notice in *D.N.B.* is in the main abstracted from Memoir in vol. 1 of *Scientific Papers.*]

THE SCIENTIFIC PAPERS OF . . . Edited by W. G. Adams and R. A. Sampson, with a Memoir by J. W. L. Glaisher. 1896, 1900

C.U.P. 2 vols. 4to, 28½ cm.

(1) liv, 502. Portrait; facsimile of MSS. (4 pp.). (2) xxxii, 646. 6 folding charts.

[Pres. J. M. Wilson, on his election as President of the Math. Assoc. 1921; inscribed . . . "at whose house in Rugby the first meeting of the founders of the Association was held on 17 Jan. 1871."

Vol. 1 contains the Memoir, Challis's 'Special Report of Proceedings in the Observatory relative to the new Planet' dated 12 Dec. 1846, and 62 published papers, mostly Astronomical. Vol. 2 contains a course of lectures on the Lunar Theory and 15 Astronomical papers, all edited from unpublished MSS. by R. A. Sampson, and papers on Terrestrial Magnetism, similarly edited by W. G. Adams and arranged to form a consecutive treatise. The *C.U.P.* published separately the 'Lectures on the Lunar Theory' in the form in which they appear in Vol. 2 above. An unsigned review of Vol. 1 appeared in *Nature.* Vol. 56, p. 73 (1897, May 27). For review of Vol. 2 by E. W. Brown *v. Bull. Amer. Math. Soc.* vii, 272. (1901, March.)

ADAMS, John Lowry.

Surely the least obstinate of cranks, if one were to judge only from the beginning of a paragraph in his first preface. "The trespasser in any carefully cultivated, and closely guarded, scientific field will be sure to find pitfalls, great and small, waiting for him on every side. The knowledge of this, and of my poor equipment, being only a raw recruit in the astronomical ranks, for any conflict with the recognised guardians of the astronomical field, led me to hope that I might be spared this ordeal; but the burden has been cast upon me and . . . I shall view the detection . . . of any slips with a reasonable degree of equanimity." But the sentence proceeds "in the sure and certain faith that my main contentions are unassailable."

This author's name is not in the British Museum Catalogue.

THE MILKY WAY. The solution of the problem of the Milky Way, shewing it to be a special shadow effect. 1905

Sydney; Turner and Henderson. iv, 5-44. 10 diagrammatic plates, 2½ cm.

The theory is that the earth's shadow is densest near the equator, and that the appearance of the Milky Way is due not to any actual multiplicity of stars but to the greater efficiency of the shadow in affording a contrast by which

stars become visible. "Those who ridicule the possibility of the earth's shadow having any effect in bringing into greater or less relief, according to its degree of darkness, the distant stars, must also, I presume, be prepared to ridicule the idea that moonlight can have any effect on our view of the stars.

THE INFINITY OF THE STARRY UNIVERSE. 1906

Sydney; Turner & Henderson. iv, 5-40. 6 diagrammatic plates (2 folding). 21½ cm.

In support of his theory of the Milky Way, because his previous book has "so far elicited no public comment" and he is obliged to conclude that his explanation is not regarded seriously. His competence is shewn in the course of an attack on the "mathematical humbug" of Struve's proof that if the stellar universe was infinite and light was not absorbed by space, there could be no dark background to the sky. The proof, quoted from the *Encyclopedia Britannica*, uses the phrase 'spherical angle': "To the uninitiated a 'spherical angle' may seem a peculiar expression in view of what is the ordinary idea of an angle, but it means 'the angle made by the meeting of two arcs of great circles which mutually cut one another on the surface of a globe or sphere.'"

ADAMS, William Grylls, 1836.

Youngest brother of the astronomer, and one of the Editors of the collected edition of his Scientific Papers.

ADDISON, Joseph, 1672-1719.

The dramatist, essayist, and poet, contributed to the later editions of Burnet's *Theory of the Earth* a Latin ode in praise of the author and an English version of the eulogy. We quote a verse:

Jamque alta cœli mœnia corruunt
Et vestra tandem pagina, (proh nefas !)
BURNETTE, vestra augebit ignes,
Heu socio peritura mundo.

And now the kindling Orbs on high
All Nature's mournful End proclaim;
When thy great WORK (alas !) must die,
And feed the rich victorious Flame:
Give Vigour to the wasting Fire
And with the World TOO SOON expire.

Manners change. It is not likely that future editions of *Problems of Cosmogony and Stellar Dynamics* will be enriched by verses on the Secretary of the Royal Society, nor does even Prof. Housman issue his poems simultaneously in the two languages. E. H. N.

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274. d'Alembert complimented a young man on his solution of a mathematical problem, "Ah!" said the young fellow, "what I desire is to be a member of the Academy": "Sir, this is just what with these dispositions you will never be. Science must be loved for her own sake, not for the attendant advantages; there is no other way of making progress in it."

275. The acquisition of the bookstalls of the London and North-Western Railway in 1849 by Mr. W. H. Smith produced a gigantic increase in the dimensions of his operations, and he invited his old schoolfellow at Tavistock, Mr. Lethbridge, to abandon the teaching of mathematics for the selling of newspapers, and by this judicious choice he acquired a partner of unusual abilities and powers of organisation.

MATHEMATICAL NOTES.

743. [B. 3. a. c.] *The Numerical Evaluation of a Resultant.*

If $a_1, a_2, \dots a_m$ are the roots of

$$a(x) \equiv a_0 x^m + a_1 x^{m-1} + \dots + a_m = 0$$

and $\beta_1, \beta_2, \dots \beta_n$ those of

$$b(x) \equiv b_0 x^n + b_1 x^{n-1} + \dots + b_n = 0,$$

the resultant of $a(x)$, $b(x)$ is

$$R \equiv a_0{}^n b_0{}^m \Pi(a_i - \beta_j), \quad i = 1, 2, \dots m, \quad j = 1, 2, \dots n,$$

which $= a_0{}^n \Pi b(a_i) = (-)^{mn} b_0{}^m \Pi a(\beta_j)$.

To evaluate R in a numerical case, let $n \geqslant m$; then if

$$b(x) = r_1(x) a(x) + c(x),$$

with $c(x) = c_0 x^p + c_1 x^{p-1} + \dots + c_p = c_0 \underset{1,p}{\Pi}(x - \gamma_u)$, and $p < m$,

$$b(a_i) = c(a_i), \quad \text{and} \quad R = a_0{}^n \Pi c(a_i) = (-)^{mp} a_0{}^{n-p} c_0{}^m \underset{1,p}{\Pi} a(\gamma_u).$$

Next put $\qquad a(x) = r_2(x) c(x) + e(x)$

with $\qquad e(x) = e_0 x^q + e_1 x^{q-1} + \dots + e_q = e_0 \Pi(x - \epsilon_v)$, and $q < p$;

then $a(\gamma_u) = e(\gamma_u)$ and

$$R = (-)^{mp} a_0{}^{n-p} c_0{}^m \Pi e(\gamma_u) = (-)^{mp+pq} a_0{}^{n-p} c_0{}^{m-q} e_0{}^p \Pi c(\epsilon_v).$$

Repeating this process, the successive degrees $n, m, p, q, \dots$ form a decreasing sequence of positive integers (each member usually exceeding its successor by a unit), and, if the last two remainders are

$$s(x) \equiv s_0 x^f + s_1 x^{f-1} + \dots + s_f, \quad t(x) = t_0 x + t_1,$$

the calculation of R is completed by noting that

$$s_0 t(\sigma_1) \dots t(\sigma_f) = s_0 t_1{}^f - s_1 t_1{}^{f-1} t_0 + \dots + (-)^f s_f t_0{}^f.$$

[The whole process is equivalent algebraically to that of going through the H.C.F. process for $a(x)$, $b(x)$, but brings out the exact numerical value of $\Pi(a_i - \beta_j)$.]

As an illustration we calculate the discriminant of the equation

$$a(x) \equiv x^5 - 3x^2 + 11 = 0.$$

If $a_1, \dots a_5$ are the roots,

$$D = \Pi(a_i - a_j)^2, \text{ and writing}$$

$$a'(x) = \Sigma(x - a_2)(x - a_3)(x - a_4)(x - a_5) = 5x^4 - 6x = 5(x - \beta_1) \dots (x - \beta_4),$$

$$\begin{aligned} D &= \Pi a'(a_i) = 5^5 . \Pi(\beta_j{}^5 - 3\beta_j{}^2 + 11) \\ &= 5^5 . \Pi\{-\tfrac{9}{5}\beta_j{}^2 + 11\} = 5 . \Pi(-9\beta_j{}^2 + 55) \\ &= 5 . 9^4 . \Pi(\beta_j - \gamma_1)(\beta_j - \gamma_2), \text{ where } \gamma^2 - \tfrac{55}{9} = 0, \\ &= 5 . 9^4, (\gamma_1{}^4 - \tfrac{6}{5}\gamma_1)(\gamma_2{}^4 - \tfrac{6}{5}\gamma_2) \\ &= 5 . 9^4 . (\tfrac{3025}{81} - \tfrac{6}{5}\gamma_1)(\tfrac{3025}{81} - \tfrac{6}{5}\gamma_2) \\ &= 5 . (3025 - \tfrac{486}{5}\gamma_1)(3025 - \tfrac{486}{5}\gamma_2) \\ &= 5 . 3025^2 - 11 . 162^2 = 45464441. \end{aligned}$$

W. E. H. BERWICK.

744. [A. 3.] *Quintics and Higher Equations.*

There being no general method of resolving the higher equations, when such are met with in the application of mathematics, it becomes necessary to employ some method of approximation to determine the roots.

Several methods are known by which the roots of such equations can be approximated theoretically to any desired degree of accuracy whatever, but in practice the difficulty of obtaining great accuracy is also very considerable.

The author has employed the following method with satisfactory results as regards time and labour considered in relation to accuracy achieved.

Any equation $\quad x^n + Ax^{n-1} + Bx^{n-2} \ldots Qx + R = 0$

may be written $\quad x^2(x^{n-2} + Ax^{n-3} + Bx^{n-4} \ldots + P) = -(Qx + R).$

Put $\quad$ (1) $\quad x^2(x^{n-2} + Ax^{n-3} + Bx^{n-4} \ldots + P) = y.$

$\quad$ (2) $\quad -(Qx + R) = z.$

Assume $x = b$, some value which inspection suggests lies in the neighbourhood of a root. Substitution in equation (1) gives a value of y, say c.

Put $z = c$ in equation (2), and calculate corresponding value of x, say d.

Substitute $x = d$ in equation (1) and again calculate y, and substitute for z in equation (2), and proceed as before.

As the process is continued the value of y will approach that of z in the previous operation.

When $y = z$ the equation will be solved.

The method requires nothing more difficult than the solution of a linear equation.

The simplest proof is a graphical one.

Fig. (1) shows the first of the two cases that arise.

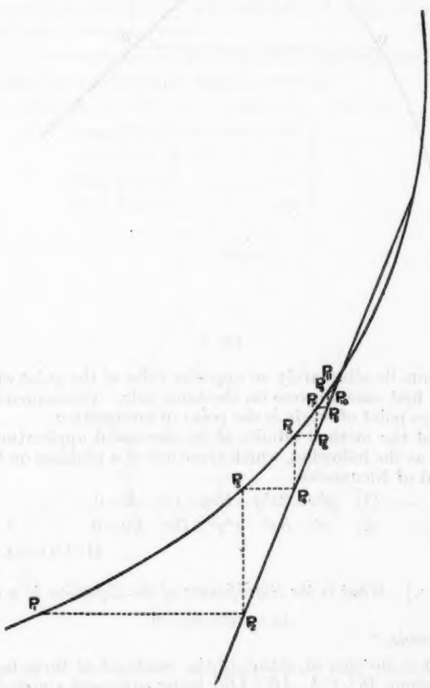


FIG. 1.

It is evident that we obtain points lying alternately on a curve and on an intersecting straight line.

Each pair of successive points has one, and only one, common ordinate. Hence the two series of points rapidly approach the point of intersection where they coincide, and both ordinates are common.

Here $$y = z.$$

Fig. (2) shows an alternative result.

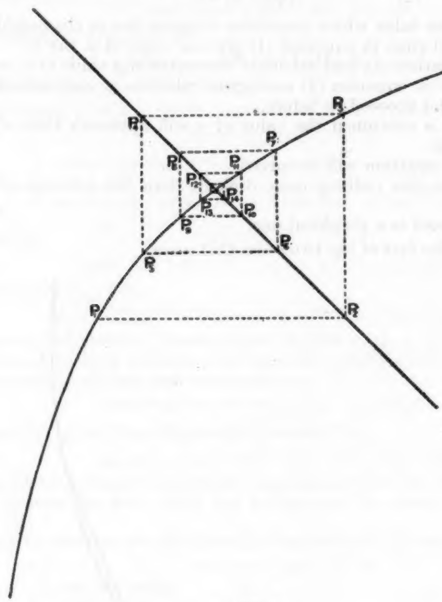


FIG. 2.

Here the points lie alternately on opposite sides of the point of intersection, whereas in the first case all were on the same side. Consequently their locus is a spiral, whose point of origin is the point of intersection.

The utility of the method admits of its successful application to complex equations such as the following, which arose out of a problem on the properties of the Conchoid of Nicomedes.

$$(1)\quad y^3x + 2x^2y + Bxy - Cx - E = 0.$$

$$(2)\quad x^3 - Bx^2 - x^2y^2 + Dx - Ey = 0.$$

H. CANSDALE, A.C.W.A.

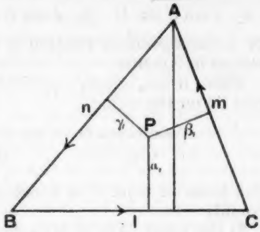
745. [V. 1. a. ι.] *What is the Significance of the Equation to a Straight*

$$l\alpha + m\beta + n\gamma = 0$$

in Trilinear Coords.?

It is this: it is the line of action of the resultant of three forces propl. to l, m, n acting along BC, CA, AB (ABC being supposed a material triangular lamina).

For obviously the moment of the system vanishes about any pt. P in the line $l\alpha+m\beta+n\gamma=0$.



What is the resultant of these forces ?

Evidently

$$\sqrt{\{(l-m\cos C-n\cos B)^2+(m\sin C-n\sin B)^2\}}$$
$$=\sqrt{\{l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C\}}$$
$$=R \text{ say.}$$

Now the moment of the system about $P(\alpha_1\beta_1\gamma_1)$ is

$$l\alpha_1+m\beta_1+n\gamma_1.$$

This must be = moment of the resultant about P

$$=R\times\text{perp. from } P \text{ on } l\alpha+m\beta+n\gamma=0;$$

$\therefore$ perp. from P on $l\alpha+m\beta+n\gamma=0$ is equal to $(l\alpha_1+m\beta_1+n\gamma_1)/R$.

Thus, from statical considerations, we get the formula in trilinears—which is not particularly easy to obtain.

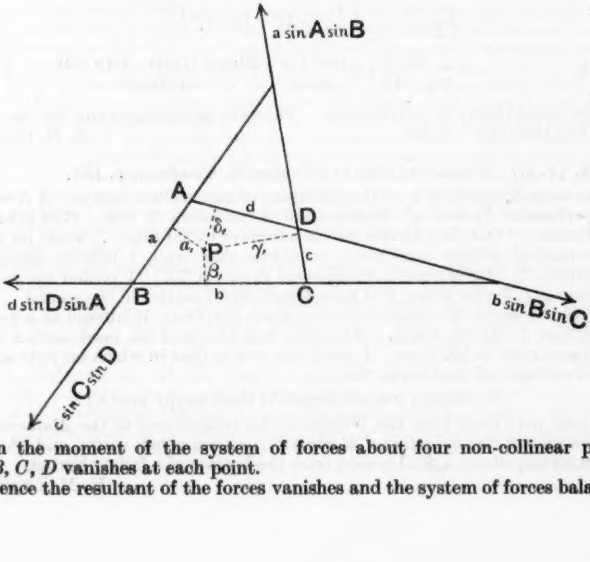
This was, I am told, one of Routh's "tips" in years gone by, and if so there must be some reader who can tell us if it ever appeared in print. I give it now because of the analogy with the following quadrilateral application, which, as far as I know, is new. R. F. Davis.

746. [V. 1. a. ι.] *Formulae derived from Statical Considerations.*

$ABCD$ is a material quadrilateral lamina (convex).

$$AB=a, \quad BC=b, \quad CD=c, \quad DA=d.$$

Forces
$$\left.\begin{array}{llll} c\sin C\sin D & \text{act along} & AB, \\ d\sin D\sin A & ,, \quad ,, & CB, \\ a\sin A\sin B & ,, \quad ,, & CD, \\ b\sin B\sin C & ,, \quad ,, & AD. \end{array}\right\}$$



Then the moment of the system of forces about four non-collinear points A, B, C, D vanishes at each point.

Hence the resultant of the forces vanishes and the system of forces balances.

Thus, if $\alpha_1, \beta_1, \gamma_1, \delta_1$ be the "quadrilinear" coordinates of ANY point P,

$$\alpha_1 . c \sin C \sin D - \beta_1 . d \sin D \sin A + \gamma_1 . a \sin A \sin B - \delta_1 . b \sin B \sin C = 0,$$

or a homogeneous relation of the first degree exists between $\alpha_1, \beta_1, \gamma_1, \delta_1$ as stated in SALMON.

Then, if $\alpha_1\alpha_2 = \beta_1\beta_2 = \gamma_1\gamma_2 = \delta_1\delta_2 =$ (semi-minor axis)^2 of inscribed conic, both foci lie on the cubic

$$\frac{c \sin C \sin D}{\alpha} - \frac{d \sin D \sin A}{\beta} + \frac{a \sin A \sin B}{\gamma} - \frac{b \sin B \sin C}{\delta} = 0,$$

the locus of a pt. P at which $A\hat{P}B$, $C\hat{P}D$ are equal or supplementary (an old friend).

If the quad. $ABCD$ is cyclic so that its opposite angles are supplementary,

$$\alpha_1 c - \beta_1 d + \gamma_1 a - \delta_1 b = 0.$$

If the quad. $ABCD$ is circumscribed to a circle there is a pt. for which $\alpha = \beta = \gamma = \delta$.

$$c \sin C \sin D - d \sin D \sin A + a \sin A \sin B - b \sin B \sin C = 0,$$

and I think this should come out equivalent to $a + c = b + d$.

R. F. DAVIS.

747. [V. a.] *Extension of Dufton's Rule.*

Divide the interval into $5n$ parts; then the area is approximately equal to the interval multiplied by the mean of the

1st, 4th, 6th, 9th, ... $(5n-4)$th and $(5n-1)$th ordinates.

If the curve be $y = a_0 + a_1x + a_2x^2 + \dots a_px^p + \dots$ the true area for the interval 0 to 1 is $\int_0^1 y\,dx = \sum_p a_p/(p+1)$, and the approximate area is

$$\sum_p \frac{a_p}{2n\,.\,(5n)^p}\left[\sum_1^n \left\{(5n-4)^p + (5n-1)^p\right\}\right]$$

or

$$\sum_p \frac{a_p}{2\,.\,n^{p+1}}\left[\sum_1^n \left\{(n-\tfrac{4}{5})^p + (n-\tfrac{1}{5})^p\right\}\right],$$

which

$$= \sum_p \frac{a_p}{p+1}\left[1 + \frac{(p+1)p}{300n^2} - \frac{29(p+1)p(p-1)(p-2)}{90000n^4} + \dots\right]$$

by the usual theory of polynomials. Thus the percentage error for the term a_p is less than $p(p+1)/3n^2$.

N. M. GIBBINS.

748. [V. 8.] *Answer to Query in Gleaning* 57, *Gazette*, x. p. 133.

The verse in question is at the beginning of an excellent course—*A New and Comprehensive System of Mathematical Institutions*, 2 vols., 1759-1764, by B. Martin. "Old *Ben Martin* (as his admirers called him)... wrote on every mathematical subject (and never otherwise than well, I believe, except on biography)." [De Morgan's *Arithmetical Books*, p. 73.] I picked up his two volumes at different times, and have found useful matter in the second. How he came to insert the comical curious poem (by Chas. Wildbore) as a recommendation I cannot think. His wine was too good to need such a bush. The poem runs to 102 lines. I think the best is that in which he puts among the advantages of the Greeks that

"No foreign tongue perplex'd their happy youth!"

The poet may have been the Wildbore who contributed to the *Mathematical Repository* and the *Miscellanea Mathematica*, or some such works, and who had a controversy about a fluid issuing from the bottom of a falling bucket.

E. M. LANGLEY.

REVIEWS.

The Reorganization of Mathematics in Secondary Education. A Report by the National Committee on Mathematical Requirements under the auspices of the Mathematical Association of America. Pp. x + 652. 1923.

Such care and labour have been devoted to the production of this report that we should be ungrateful if we did not commend its many excellences before criticising, with reluctance, its unfortunately conspicuous retrograde aspects. The report, like so many of its kind, is eclectic, and draws its facts and ideas from many sources, but to the carefully selective reader it yields much useful information and many pregnant suggestions. It is, on the whole, progressive in spirit, and it is certainly rich in detail. No less than 569 articles or books are listed in the Bibliography of the Teaching of Mathematics (for the years 1911 to 1921); and in the long chapter on tests there is a list of 165 books or papers on the subject of mathematical tests alone. This kind of thing enhances the value of the report as a book of reference, but the bibliographies (and the tables of researches in the pocket of the cover) would have been the better for a little pruning; not everybody's contribution to the literature of a subject is of the same value. Also the report might have avoided in places what looks like a mere pretence of thoroughness, as, for example, in the chapters on curriculum and teacher training in foreign countries, which might well have been omitted.

The account in the early chapters of the aims of mathematical teaching is unexceptionably sane—it is too reasonable to be convincing! The American mind appears to delight in clear-cut statements and is often content with ideas that are superficial if only they are clear. We, therefore, find rather more general talk about mathematical teaching than the practical teacher is likely to desire or to need. Those who like the broad statement that we all admit, and none of us require, will welcome the report's assertion that "The primary purposes of the teaching of mathematics should be to develop those powers of understanding and of analyzing relations of quantity and of space which are necessary to an insight into and control over our environment and to an appreciation of the progress of civilization in its various aspects, and to develop those habits of thought and of action which will make these powers effective in the life of the individual." In justice to the Committee we hasten to add that in the details of curriculum and procedure their report does its best to keep abreast of the high aims set out in the first few chapters. It was a good plan to devote a chapter to Relationships in Mathematics, but this chapter would have been somewhat more effective if it had been less general and had included more specific illustrations of the ways in which the notion of relationship could be introduced into the school course. The chapter on Disciplinary Values in Education is as backboneless as discussions of that topic are apt to be; perhaps the debilitating cautiousness of this chapter is due to the committee's anticipation of captiousness on the part of psychologists. The teacher who will take the trouble to make a careful selection of the suggestions made in the chapter on experiments in schools may find that the most useful part of the report.

We turn to the longest chapter in the report, that by Mr. C. B. Upton on Standardized Tests in Mathematics for Secondary Schools. This chapter contains all that we want to know about what has been done with mathematical tests in America. It has, however, this much in common with the report on tests that our own Board of Education has recently issued: it leaves unanswered most of the questions that come to our mind when we consider the testing movement as a whole. Specimens of such questions are:

(*a*) Can a test which includes such a variety of problems that it suits the ideas of all teachers be regarded as measuring any definite ability? (*b*) How much truth is there in the assumption, made by all authors of mathematical tests, that there is a rigid distinction between "mechanical processes" and "reasoning"? (*c*) How can we ensure that the introduction of standardised tests will not tend to produce a uniformity of curriculum and aim that would

cripple the best efforts of teachers? Not one of those questions does this report face fairly and squarely; yet it is idle to carry on the making of tests until we have, as the French say, orientated ourselves with regard to those, and to many other, questions.

Mr. Upton has made a very thorough survey of the various tests that have been published, and his very thoroughness appears to us a defect. He half sanctions (by describing the tests or by reproducing them in his pages) many tests of which he evidently disapproves heartily. Thus he gives specimens of tests (we suggest that the Rugg-Clark and the Hotz tests in algebra are instances) which are thoroughly unsound pedagogically and point in a direction directly opposed to that followed by good modern teachers, and he contents himself generously with such lukewarm criticism as this: "The main danger of the present algebra tests, therefore, is that they may tend to give teachers the impression that the topics that are being tested are the all-important ones, and thus they tend to direct emphasis upon types of problems that are relatively unimportant." After honouring the Rugg-Clark tests with 15 pages of his chapter he says: ". . . the problems of the Rugg-Clark Scale were based upon an analysis of the existing textbooks, which is not the best source of material for reform purposes." He has an especially suggestive remark which is perhaps the best thing he says: "It is my personal belief, however, that a great deal of this problem-solving attitude can also come from the right method of handling and thinking about the fundamental operations themselves"; that is, a so-called "mechanical process" *can* be made as much a problem as can an exercise with "concrete" data. We recommend this contribution of Mr. Upton's as a good starting point for that constructively critical discussion of the whole problem of standard tests that is so urgently needed in this country.

E. R. HAMILTON.

Œuvres de G. H. Halphen. Tomes: I, xliv + 570, 40 fr., 1916; II, viii + 560, 64 fr., 1918; III, xii + 518, 90 fr., 1921; IV, xvi + 657, 100 fr., 1924. (Gauthier-Villars.)

Georges Henri Halphen, soldier and mathematician, was born in 1844. Courageous and enthusiastic in his profession, he was decorated on the battlefield in the war of 1870, and withdrew from the academic circles of Paris in 1886, the year in which he published the first volume of a *Traité des Fonctions Elliptiques*, to take over by his own desire an active military command, having in the meantime won the highest honours from the Academies of Paris and Berlin. Thenceforth he was doubly occupied, for the time which was not absorbed by his duties was devoted to his treatise. The second masterly volume appeared in 1888, but to the grief and dismay of the mathematical world the writer, weakened perhaps by his double toil, succumbed to a sudden attack of illness in the following spring, and died at the early age of forty-four. The French Academy undertook the publication of whatever should be found among his papers in continuation of the treatise, and a slender volume of *Fragments* was issued in 1891.

Some twenty years afterwards, Mme. Halphen determined to raise to her husband a worthy monument, by assembling his writings, other than the *Traité*, in a collected edition. The responsibility of supervision was accepted by Jordan, Poincaré, and Picard; this brilliant collaboration affords in itself more striking evidence of the value attached to Halphen's work by those competent to judge it than even the eloquent words of Picard's oration to the Academy—"Mettant à profit, avec un art consommé, le secours que peuvent se prêter les diverses parties des mathématiques, il a su pousser jusqu'à leur dernier terme les solutions des problèmes qu'il s'est posés"—of Poincaré's notice in the *Journal de l'Ecole Polytechnique*, which, speaking of the absence of any notes on some problems which he was known to have solved, concluded: "Perte irréparable jusqu'à ce qu'il naisse un autre Halphen," or of Jordan's tribute in *Liouville*: "Mais il ne mourra pas tout entier; les œuvres qu'il a laissées feront vivre son nom tant qu'il y aura des mathématiques."

The collected edition is in four volumes. The papers are in order of publication, but since Halphen's practice was to exhaust one field of investigation at

a time, this order actually implies a considerable measure of continuity. The important memoirs in the first volume are on the characteristics of systems of conics, on the singularities of plane curves, and on points at which an algebraic curve satisfies a given differential equation. The second volume contains papers on the singularities of twisted curves and of algebraic surfaces, on the differential invariants of plane and twisted curves, and on representation in series. In the third volume are the famous memoirs which gained the awards of Paris and Berlin, the one on the integration of linear differential equations, the other on the classification of algebraic twisted curves. The concluding volume has just reached us, and what follows relates only to this volume.

The longest single memoir (93 pp.) is of special interest to the English reader: it is the study of singularities which was published in 1884 as an appendix to the French translation of Salmon's *Higher Plane Curves*, and it incorporates the work which the author had done on the subject ten years earlier. (By the way, is it news to any of our readers who have despaired of coming across a copy of Salmon's book in the original English, that this translation is still obtainable? The German edition is out of print.) Of the four memoirs next in length, the substance of two (63 pp. and 69 pp.) on certain applications of elliptic functions to mechanics was absorbed by the author into the second volume of the *Traité*, and one (41 pp.) on complex multiplication of elliptic functions, with special reference to $\sqrt{(-23)}$, was reprinted among the *Fragments* in the third volume. Only one long memoir (68 pp.) remains to be now for the first time accessible. This, from *Liouville*, 1885, under the uninformative title *Sur un problème concernant les équations différentielles linéaires*, discusses the utilisation for the purposes of integration of the knowledge of a definite algebraic relation between the unknown solutions. The general theory is followed by applications in detail to equations of the second and third orders, and it is in discussing a particular equation of the second order under the assumption that the product of four solutions is to be a polynomial in the independent variable that Halphen discovers the relation of Lamé's equation for integral values of $2n$ to the algebraic problem "Given a biquadratic binary form Φ, find a binary form Z, such that the covariant $\Phi_{22}Z_{11}-2\Phi_{12}Z_{12}+\Phi_{11}Z_{22}$ is identical with Z," and so explains for the first time the similarity of the solution for odd values of $2n$ to the solution for integral values of n.

One of the shorter reprinted papers calls for special mention. The student who is expert in any of the subjects which Halphen made peculiarly his own will need no help to find matter for enjoyment as soon as the book is in his hands. The general reader, to see how the touch of a master can transmute the commonest material, could hardly do better than turn to p. 276 and follow the deduction of the whole theory of the solution of equations of the third and fourth degrees from the simple observation that if f and ϕ are polynomials of the same degree n, the expression $f\phi^{(n)}-f'\phi^{(n-1)}+f''\phi^{(n-2)}-\dots$, since its derivative is obviously zero, is itself a constant.

About one quarter of this fourth volume consists of unpublished papers, all quite short, which naturally deal with a variety of topics, and it is with curiosity that one approaches these. Halphen's papers were examined for the Academy by a committee composed of Appell, Darboux, Picard, and Poincaré, and a report by Poincaré, dated 1901, on the manuscripts which this committee thought worthy of preservation, introduces the memoirs to which it refers. Perhaps the most surprising of these is one to which the editors attach the date 1875, on the differential geometry of a surface; it was characteristic of the author that as he did not devote time to following the ideas up to a point of relative finality, the work did not see the light. A variable point of the surface is referred to two tangents and the normal at a point M, and the surface is then regarded as determined by the expression of z as a function of x and y, but the point M itself is allowed to vary, and Halphen calculates the consequent changes in the partial derivatives of z of the first three orders. Of the results obtained, a few, including one on Weingarten surfaces, were new and were announced at the time without proof, but most were recognised as familiar; the interest of the paper is not in the individual results but in their coordination.

After the unpublished papers is an extract whose insertion is explained by a letter from Jordan : " Ne conviendrait-il pas de reproduire comme fragment posthume [ces] pages de mon *Cours d'Analyse* ? Elles ont trait à l'intégration très intéressante d'une classe d'équations différentielles ; et bien que la rédaction soit de moi, la découverte du théorème et *toute* la démonstration sont de [Halphen]." Could courtesy be more perfect ?

The hope which the editors formerly expressed of finding mathematical correspondence has not been fulfilled, and only some extracts from letters to Zeuthen are printed. Finally there is a classified table of the contents of the four volumes.

A word of praise is due to M. Vessiot, on whose shoulders the routine work of the publication has fallen. It is a melancholy thought that of the three editors to whom Mme. Halphen confided the enterprise only one survives to see its successful completion. To M. Picard, who was engaged on this tribute to another great fellow-countryman even before his single-handed exploit of editing the works of Hermite was fully accomplished, we offer our very warm congratulations. E. H. N.

Analytical Mechanics, Comprising the Kinetics and Statics of Solids and Fluids. By E. H. Barton, F.R.S. 2nd edition. Pp. 593. 21s. net. 1924. (Longmans.)

The ground covered by this book is very extensive, as the 483 pages of text contain not only the ordinary two dimensional Statics and Kinetics, but Solid Rigid Kinetics, including gyroscopic motion, as well as chapters on Strains, Attractions, Hydrostatics, Hydrokinetics and Elasticity. There are also 70 pages of miscellaneous examples, mostly taken from examination papers of the London University.

The greater part of the book consists of three main sections—Kinematics, to which about 170 pages are devoted, including a chapter on Strains ; Kinetics, about 100 pages, and Statics, about 120 pages.

The book requires a fair knowledge of the Calculus, and it is hardly likely that anyone will tackle it who has not previously read elementary treatises on the subject.

The reader will quickly realise the power of the Calculus in dealing with problems on motion, and should he have carried away the impression so commonly derived from elementary books on mechanics that s is always equal to $ut+\frac{1}{2}ft^2$, the excellent set of problems solved in the chapter on Rectilinear Motions will quickly drive that idea from his mind. The next two chapters give a good idea of two dimensional kinematics, and the subject matter is divided up so conveniently among the chapters that, by judicious skipping, a two-dimensional course may easily be mapped out by anyone who wishes to avoid the real difficulties of mechanics, viz. three-dimensional motion. In such a course the student will find all the well-known problems, such as the pendulum and rolling bodies, as well as some less popular ones, *e.g.* snow sliding on a roof, and bodies which slide as well as roll.

The three-dimensional chapters are necessarily difficult and there is hard reading in Chapters VIII., XIV. and XX. In this part of the subject the student requires much practice, and the two latter chapters, particularly, would be greatly improved in value by the addition of a considerable number of problems, *e.g.* on the rolling of spheres on rough planes, as suggested by the author himself on p. 301. There are many examples in the text throughout the book, but less than there appear to be, for a very large number of them are merely directions to write out the preceding bookwork.

The chapters referred to give the impression that the author is attempting to compress rather too much into the book, and this is particularly noticeable in Chapter XX. on Hydrokinetics, where the reader is given practically nothing on which to exercise his ingenuity, though there are one or two examples on p. 499. The collection of miscellaneous questions has undoubtedly improved the value of the book, and it is conveniently divided up into sections by subjects.

A few misprints were noticed on pp. 107-9, where the dots in $\ddot{\theta}$ are omitted.

Mention must not be omitted of the admirable discussion of Newton's Laws in Chapter XI., and the numerous references throughout the book to more extended treatises and to papers on the subjects discussed. In conclusion it may be said that the book is excellent for the student who, having a little mathematical skill, desires a sound knowledge of the simple problems of mechanics and some inkling of the more difficult problems as well as an insight into the theory on which is based the practical subject of strength of materials.
W. M. R.

An Elementary Course in Analytical Geometry. By The Rev. B. C. Molony. Pp. 1-167, with Answers. Parts I. and II., 3s. 6d. Part I. separately, 2s. 6d. 1924. (Bell and Sons.)

This book is evidently the outcome of experience in teaching boys who prepare for the Army examination. It covers the ground which is now fairly well marked out in several books answering this purpose. The subject is carefully graded and everywhere there is an abundance of exercises for the diligent student.

Part. I. contains such matter as the Army examination requires, and no more: Part II. supplements this with further treatment of conics and other curves. The aim in the book is not to discuss the general quadratic in x and y as illustrated by conics, but to deal with all curves whose equations are relatively simple.

The book would be improved with the addition of an index and also of a chapter on solid geometry. This would fall in quite naturally with the general aim of the book. Also the book is good enough to make one regret that in certain details it may encourage slovenliness of thought or expression among its youthful readers. There are, for instance, unsightly abbreviations everywhere: "p^t" for point; "$= \therefore x \cos a + y \sin a - p$"; "by Pythagoras." Some of the figures are overburdened, as on p. 9.

One may commend the book and it should prove useful.

College Algebra. By Lewis Parker Siceloff and David Eugene Smith. Pp. 1-246, with Tables and Index; no Answers. $1.80. 1924. (Ginn and Company.)

This book is written for first year students at colleges and technical schools in America, and is in many respects parallel with the recent book on Analytical Geometry by the same authors. The book opens with a rapid sketch of elementary algebra. The later chapters are devoted to "College Algebra"—*e.g.* mathematical induction, permutations, probability, determinants, theory of equations—followed by "Optional Special Topics"—partial fractions, interest and annuities, infinite series, and so on.

Considering the elementary nature of the work this variety of material is good. As may be expected, the treatment by these authors is lucid and, with one exception, satisfactory. Thus the four pages (43-46) on the theory of indices are logical and clear, and go to the heart of the matter.

The exception is the treatment of infinite series. The early statement of the meaning of the sum of an infinite geometrical progression is too summary to be of any use without further explanation. This would not matter so much if it was not used on p. 227 as a buttress of a more serious argument, dealing in general with infinite series. How would the authors proceed to convince their students that the limit of $(\cdot 999999)^n$ is zero when $n \to \infty$? The fact is, that the authors are here dealing in a quasi-rigorous fashion with definitions and theorems about limits which cannot be rigorously established in such an elementary book. Far better to explain that certain propositions *can* be proved, to say *where* they are proved, and to say that they are here assumed to be true, than to suggest to the inexperienced that the last word has here been said.
H. W. Turnbull.

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276. If I were teaching them to form battalia by extracting the square root . . what return could I expect for communicating this golden secret of military tactic, except it may be a dirk in my wame. . . .?—Captain Dalgetty *loq.*, *A Legend of Montrose*, c. iii.

THE LIBRARY.

160 Castle Hill, Reading.

ADDITIONS.

To the list of books given by **Mrs. Godfrey** should be added :

C. L. Dodgson New Theory of Parallels - - - - - - 1888

The gift included also pamphlets by A. S. Eddington, W. S. Franklin, B. MacNutt and R. L. Charles, G. B. Halsted, M. Pieri, M. Planck, J. G. Thomson.

It will be noticed that odd volumes appear under a few names in the list : Allcock, Appell, Bardey, Castle, Clifford, Durell, Fisher and Schwatt, Haussner, Henrici and Treutlein, Lesser, Leyssenne, Loria. If any members can fill the gaps, their generosity will be acknowledged gratefully. Still more is it to be hoped that the collection of publications of the International Commission will be gradually completed, for these are works which the Library is expected to possess. The *Cambridge Tracts* and the *Edinburgh Tracts*, like *Scientia* and Ostwald's *Klassiker*, are specially valuable to this Library, and the Librarian urges members who come across odd numbers in any of these series to remember the needs of the Association . Of the *Cours complet de mathématiques élémentaires* which Darboux edited, only three volumes are now lacking, namely, Bourlet's *Trigonométrie*, Koenigs' *Mécanique*, and Tisserand and Andoyer's *Cosmographie*.

Occasion may be taken to emphasise that funds of the Association must be available for upkeep and equipment and can not be spent as freely as we should all like, on actual books. A gift such as Mrs. Godfrey's involves incidental expenses for carriage and cataloguing, cases to accommodate three hundred books are not a negligible item, and if such works as Goursat's *Cours d'Analyse* and Klein's *Nicht-Euklidische Geometrie* come to us unbound, the least we can do is to bind them adequately.

The Librarian reports gifts as follows :

From Mr. **W. B. Allcock,** six early numbers of the *Gazette*.

From **Mrs. W. R. Bradley,** in memory of her late husband, who joined the A.I.G.T. in 1884 :

H. Briggs and
H. Gellibrand Trigonometria Britannica - - - - - - - 1633

Includes the first tables of logarithmic sines (14 places) and tangents (10 places), at hundredths of a degree, calculated by Briggs. The modern mathematician is far from agreeing with the sentiment expressed on the title-page in a quotation from Vieta : " Ex angulis latera, vel ex lateribus angulos, et mixtim in triangulis tam planis quam sphericis, assequi, summa gloria mathematici est." But a soberer estimate would hardly have sustained the pioneer throughout the immense labours necessary to produce the work which was described by Hutton as " perhaps the greatest of this kind, all things considered, that ever was executed by one person "

C. Hutton Course of Mathematics (2 vols.) {11 : O. Gregory} 1836, 1837
Mathematical Tables {6} - - - - - - 1822

D. Lardner Euclid I-VI {6, *i.e.* 4 ( ) ster.} - - - - 1838

I. G. Pardies Short, but yet Plain Elements of Geometry.
Translated from French by J. Harris {7, *i.e.* rep.} 1734

R. Potts Euclid I-VI, XI-XII - - - - - - - 1845

Includes a historical introduction which was not reproduced in the second edition

N. Roe and
E. Wingate Tabulae Logarithmicae - - - - - - - - 1633

T. SIMPSON	Doctrine and Application of Fluxions (2 vols.) {}	1823
R. SIMSON	Euclid I-VI, XI-XII {5} - - - - - -	1775

From Mr. **W. J. Greenstreet:**

E. BEUTEL	Algebraische Kurven I - - (Göschen 435) An odd volume	1909
E. R. BRESLICH	Second-Year Mathematics for Secondary Schools {2}	1916
	Third-Year Mathematics for Secondary Schools -	1917
	Correlated Mathematics for Junior Colleges - - The initial volume, *First-Year Mathematics for Secondary Schools*, is wanted to complete this set	1919
J. BRUCE	Memoir of Charles Hutton - - - - - -	1823
L. P. EISENHART	Transformations of Surfaces - - - - - -	1923
A. R. FORSYTH	Differential Equations {3} - - - - - -	1903
	Theorie der Differentialgleichungen I: Exacte Gleichungen und das Pfaff'sche Problem - - Translated into German by H. Maser ALL PUBLISHED of this translation	1893
G. H. HARDY	Integration of Functions of a Single Variable {2} (Cambridge Tracts 2)	1916
E. F. JOURDAIN	Theory of the Infinite in Modern Thought - -	1911
J. G. LEATHEM	Symmetrical Optical Instrument (Cambridge Tracts 8)	1908
G. LORIA	Guida allo Studio della Storia delle Matematiche	1916
F. S. MACAULAY	Algebraic Theory of Modular Systems (Cambridge Tracts 19)	1916
E. W. NICHOLS	Differential and Integral Calculus - - - - -	1900
H. PARKHURST	Education on the Dalton Plan - - - - - The text of this book has no interest that is specifically mathematical, but there is an Introduction by an Ex-President of the M.A., Prof. T. P. Nunn, and the Sample Assignments include some in Mathematics and Mechanics	1922
A. RIST	La Philosophie Naturelle Intégrale I - - - - Has any member a continuation of this curious work to give ?	1904
F. RUDIO	Der Bericht des Simplicius über die Quadraturen des Antiphon und des Hippokrates - - - Greek text with translation into German and historical notes. "Finally Rudio has summed up the whole controversy": Heath	1907
STREATHAM SCHOOL STAFF	Dalton Plan Assignments II: Mathematics and Science - - - - - - - - - -	1922
H. WIELEITNER	Spezielle Ebene Kurven - - (Schubert 56)	1908

Also the following Report, of the National Committee on Mathematical Requirements, under the auspices of the Mathematical Association of America:

The Reorganization of Mathematics in Secondary Education - 1923

From Mr. **W. J. Greenstreet** and Prof. **E. H. Neville:**

C. HERMITE	Œuvres (4 vols.) - - The portrait-frontispiece to one of these volumes was missing, and Mm. GAUTHIER-VILLARS have most courteously replaced it	1905, 1908, 1912, 1917

From Col. **R. L. Hippisley :**

E. P. Adams and
R. L. Hippisley Smithsonian Mathematical Formulae and Tables of Elliptic Functions - - - - - - - 1922
With contribution by Sir G. Greenhill.

From Mr. **T. W. Hope :**

A. De Morgan Arithmetic {6, rep. of 5 (1846)} - - - - - 1876

H. Godfray Astronomy - - - - - - - - - - - 1866

From Mr. **E. M. Langley :**

M. Brückner Uber die gleicheckig-gleichflächigen nichtkonvexen Polyeder - - - - - - - - - - - 1906
Mr. Langley expresses the hope that someone will complete his gift, by sending the volume to which this is a sequel: Brückner's *Vielecke und Vielflache*

From Mr. **P. Mamlock :** A collection of recent numbers of the *Gazette*.

From Rev. **J. J. Milne :**

W. B. Frankland Euclid I with Commentary - - - - - - 1905

Also papers by W. K. Clifford, A. Lodge, H. Perigal, and J. A. Third, the Report of the B.A. Committee on Tables for 1893 (tables of the function $-iJ_1(ix)$), and several numbers of the *Library of Useful Knowledge* as originally issued.

From Mr. **F. J. W. Whipple :**

(1) Two text-books :

R. G. Blaine The Calculus and its Applications - - - - 1909

J. Wood Algebra {13 : T. Lund} - - - - - - - 1848

(2) A pseudonymous curiosity :

Fresh Light on the Dynamic Action and Ponderosity of Matter - 1891
By 'Waterdale'. "My primary reason for gravity is *mutual shelter* to opposite wave energy afforded by two spheres or bodies"

(3) A book of which he was part-author :

C. S. Jackson,
F. J. W. Whipple,
and L. Roberts A Twentieth Century Arithmetic {4 : G. T. Locke} 1924

(4) Other school-books by W. M. Baker and A. A. Bourne (2), W. S. Beard, W. G. Borchardt (2), R. C. Bridgett and W. Hyslop, C. Davison, W. J. Dobbs, F. Gorse, W. E. Harrison, W. A. Knight, A. E. Layng, W. E. Paterson, W. J. Potter, A. J. Pressland, H. L. Reed, F. W. Sanderson and G. W. Brewster, J. H. Smith, E. Young.

Sir Thomas Muir, who has for many years been a good friend to the Library, continues to send copies of his papers, and offprints have been received also from Mr. J. Brill (2) and Prof. D. K. Picken (2).

A large collection of books is on its way to the Library from Prof. R. W. Genese ; details will be published as soon as possible.

Bulletin of the Calcutta Mathematical Society.

Journal of the Indian Mathematical Society.

The Societies to which these periodicals belong have very generously filled the gaps in our runs, and our sets are now complete from the foundation of the *Journal* in February, 1909, and of the *Bulletin* two months later.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
AT THE UNIVERSITY PRESS, GLASGOW.

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THE MATHEMATICAL ASSOCIATION.

(An Association of Teachers and Students of Elementary Mathematics.)

"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament thereunto."—BACON.

THE MATHEMATICAL ASSOCIATION, which was founded in 1871, as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

Its purpose is to form a strong combination of all persons who are interested in promoting good methods of teaching mathematics. The Association has already been largely successful in this direction. It has become a recognised authority in its own department, and has exerted an important influence on methods of examination.

The Annual Meeting of the Association is held in January. Other Meetings are held when desired. At these Meetings papers on elementary mathematics are read and discussed.

Branches of the Association have been formed in London, Bangor, Yorkshire, Bristol, Manchester, Cardiff, Sydney (New South Wales), and Queensland (Brisbane). Further information concerning these branches can be obtained from the Honorary Secretaries of the Association.

"The Mathematical Gazette" (published by Messrs. G. BELL & SONS, LTD.) is the organ of the Association. It is issued at least six times a year. The price per copy (to non-members) is usually 2s. 6d. each. The *Gazette* contains—

(1) ARTICLES, mainly on subjects within the scope of elementary mathematics;

(2) NOTES, generally with reference to shorter and more elegant methods than those in current text-books;

(3) REVIEWS, written by men of eminence in the subject of which they treat. They deal with the more important English and Foreign publications, and their aim, where possible, is to dwell on the general development of the subject, as well as upon the part played therein by the book under notice;

(4) SHORT NOTICES of books not specially dealt with in the REVIEWS;

(5) QUERIES AND ANSWERS, on mathematical topics of a general character.